Mathematical intuitions and their cerebral bases

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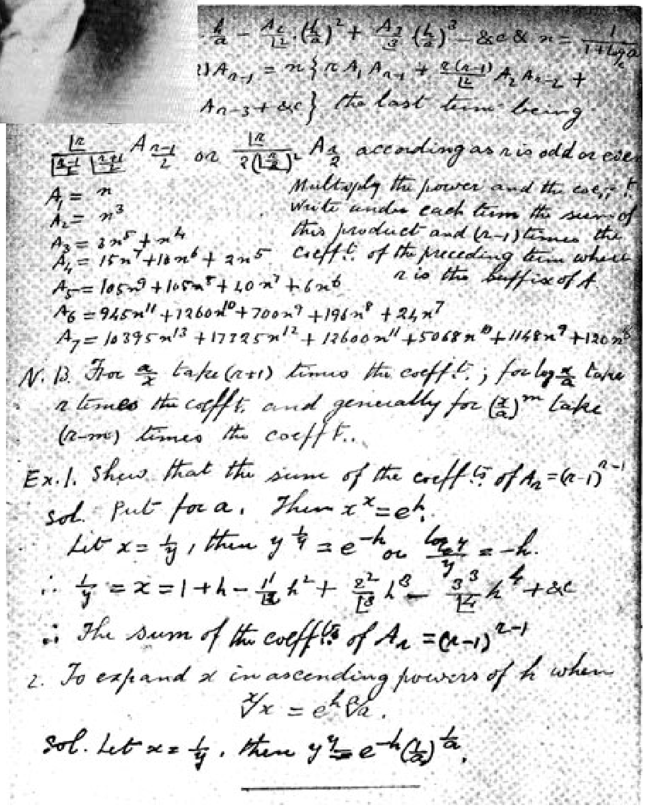
www.unicog.org

Raoul Haussmann , l ' esprit de notre temps , tête mécanique " 1919 - 1920 ,
Paris , musée national d ' Art Moderne ,

Two mathematicians

Srinivasa Ramanujan
(1887-1920)

$$\frac{1}{\pi} = \frac{2\sqrt{2}}{9801} \sum_{k=0}^{\infty} \frac{(4k)!(1103 + 26390k)}{(k!)^4 396^{4k}}$$



$A_1 = n$
$A_2 = n^3$
$A_3 = 3n^5 + n^4$
$A_4 = 15n^7 + 18n^6 + 2n^5$
$A_5 = 105n^9 + 105n^8 + 40n^7 + 6n^6$
$A_6 = 945n^{11} + 1260n^{10} + 700n^9 + 196n^8 + 24n^7$
$A_7 = 10395n^{13} + 17325n^{12} + 12600n^{11} + 5068n^{10} + 1148n^9 + 120n^8$

Multiply the power and the coeff. Write under each term the sum of this product and $(n-1)$ times the coeff. of the preceding term where n is the suffix of A.

N. B. For $\frac{a}{x}$ take $(n+1)$ times the coefft.; for $\log\frac{x}{a}$ take n times the coefft. and generally for $(\frac{x}{a})^m$ take $(n-m)$ times the coefft..

Ex. 1. Show that the sum of the coeffts of $A_n = (n-1)^{n-1}$

Sol. Put for a. Then $x^x = e^h$.

Let $x = \frac{1}{y}$, then $y^{\frac{1}{y}} = e^{-h}$ or $\frac{\log y}{y} = -h$.

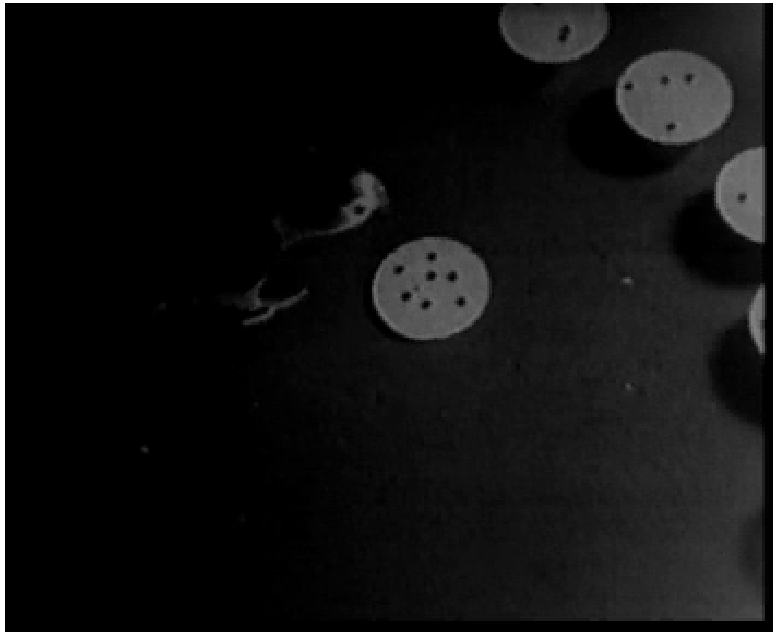
$\therefore \frac{1}{y} = x = 1 + h - \frac{1}{\underline{|2}} h^2 + \frac{2^2}{\underline{|3}} h^3 - \frac{3^3}{\underline{|4}} h^4 + \&c$

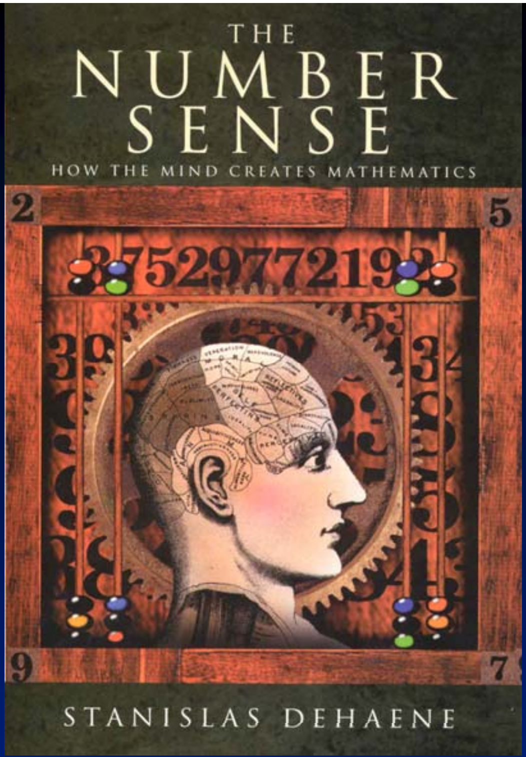
$\therefore$ The sum of the coeffts of $A_n = (n-1)^{n-1}$

2. To expand x in ascending powers of h when $\sqrt[x]{x} = e^h \sqrt[a]{a}$.

Sol. Let $x = \frac{1}{y}$, then ...

Otto Köhler's parrott (ca. 1955)



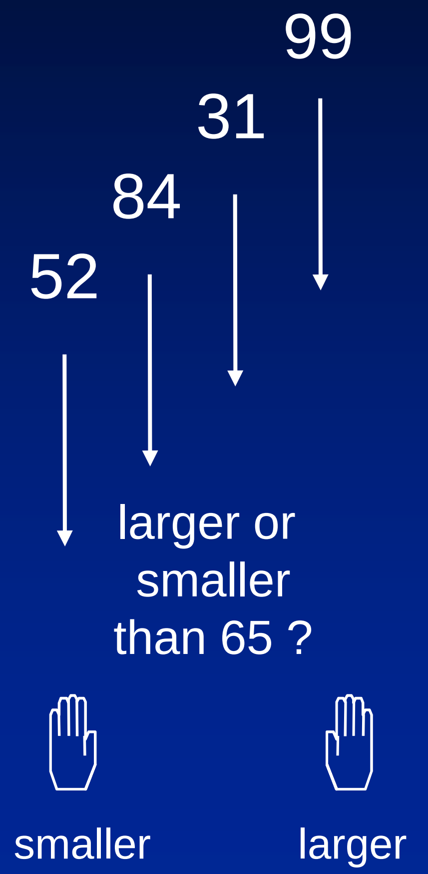


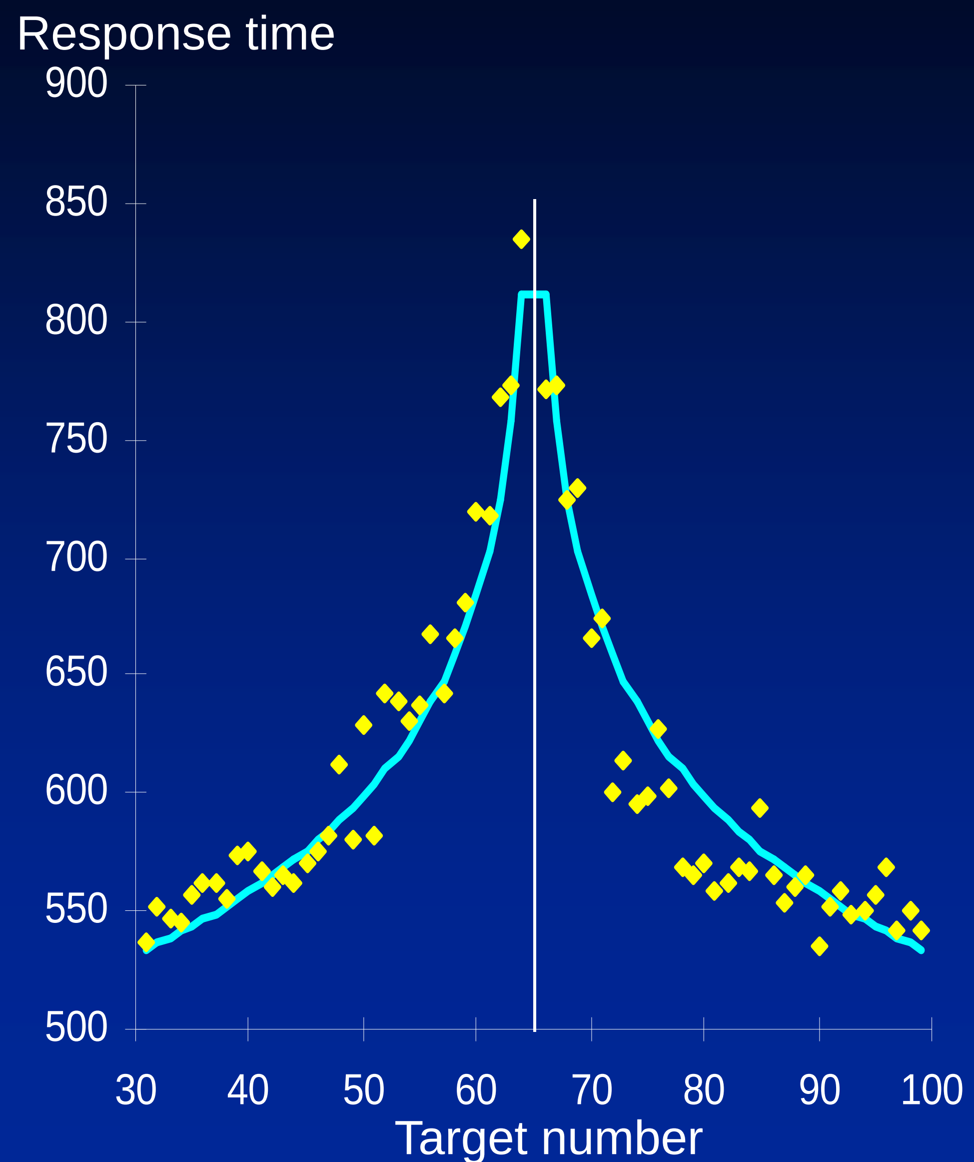
The number sense hypothesis and the foundation of mathematics

- During its evolution, our brain has been endowed with elementary representations that are adequate to certain aspects of the external world.
- These internalized representations of space, time and number, shared with many animal species, provide the foundations of mathematic intuitions.
- The cultural construction of mathematics can be seen as a search for coherence amongst these internal representations
- In particular, we are all born with a « number sense », a specific ability to represent the approximate number of objects in concrete sets, and to combine these numbers into simple operations
- Acquisition of number symbols is founded on this pre-existing number sense: we learn to connect symbols to approximate quantities.
- Arithmetic « recycles » parietal lobe circuits for numerosity perception and for spatial transformations (neuronal recycling model)

The Distance Effect in number comparison

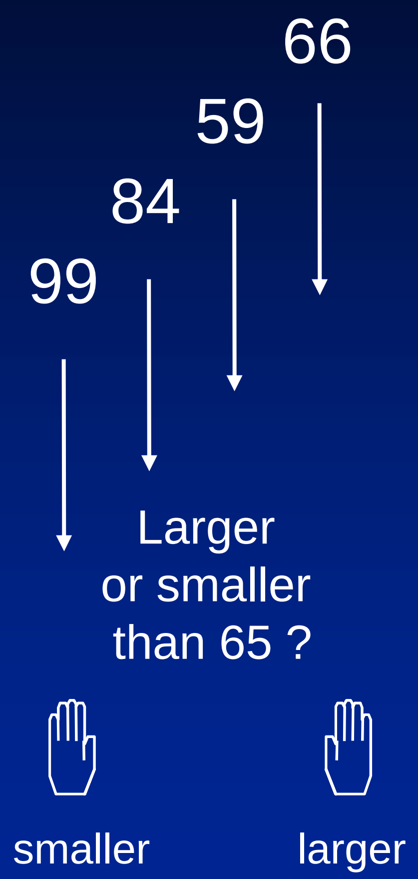
(first discovered by Moyer and Landauer, 1967)

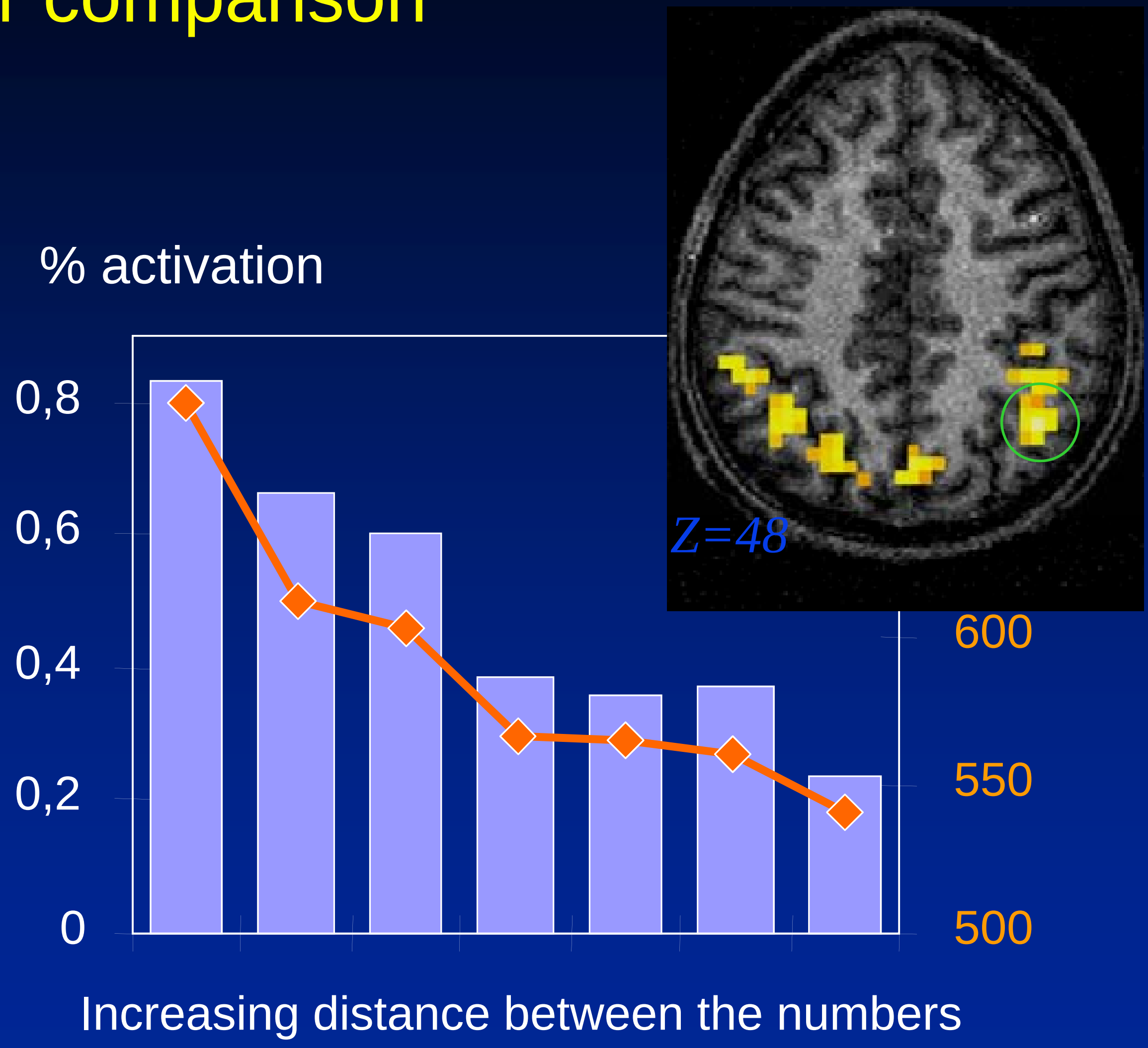




Dehaene, S., Dupoux, E., & Mehler, J. (1990). *Journal of Experimental Psychology: Human Perception and Performance, 16*, 626-641.

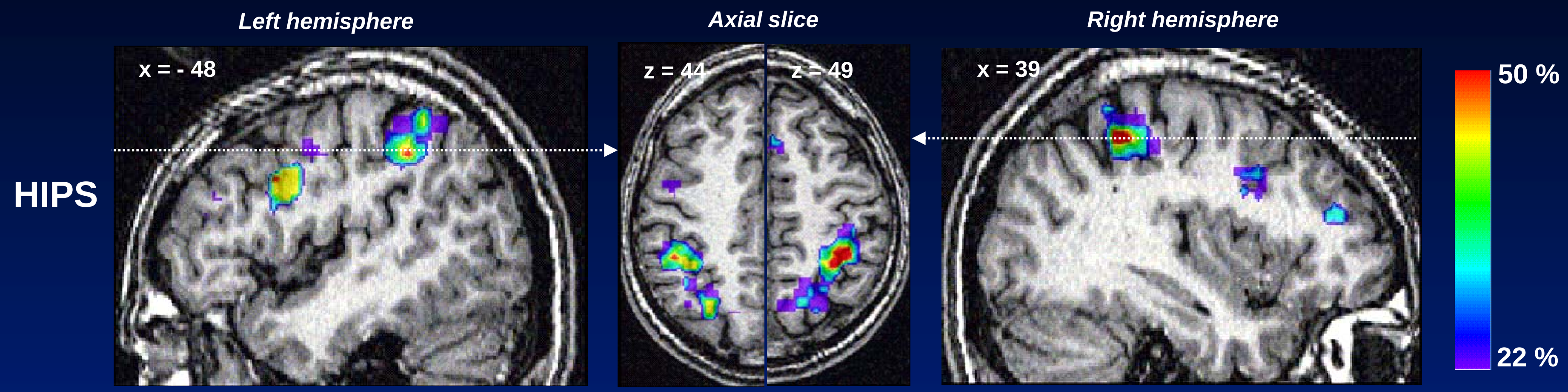
Neural bases of the distance effect in symbolic number comparison





Pinel, P., Dehaene, S., Riviere, D., & LeBihan, D. (2001). *Neuroimage, 14*(5), 1013-1026.

Number sense and the horizontal segment of the intraparietal sulcus (HIPS)



• All numerical tasks activate this region
(e.g. addition, subtraction, comparison, approximation, digit detection…)

• This region fulfils two criteria for a semantic-level representation:
-It responds to number in various formats (Arabic digits, written or spoken words), more than to other categories of objects (e.g. letters, colors, animals…)
-Its activation varies according to a semantic metric (numerical distance, number size)

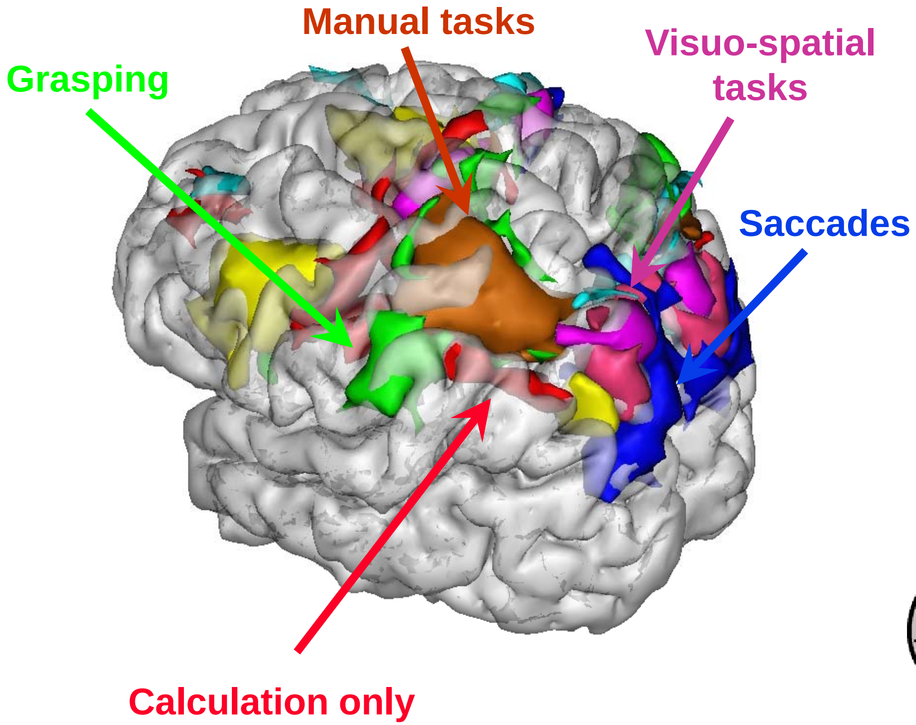
Dehaene, S., Piazza, M., Pinel, P., & Cohen, L. (2003). *Cognitive Neuropsychology*

Hypothesis 1

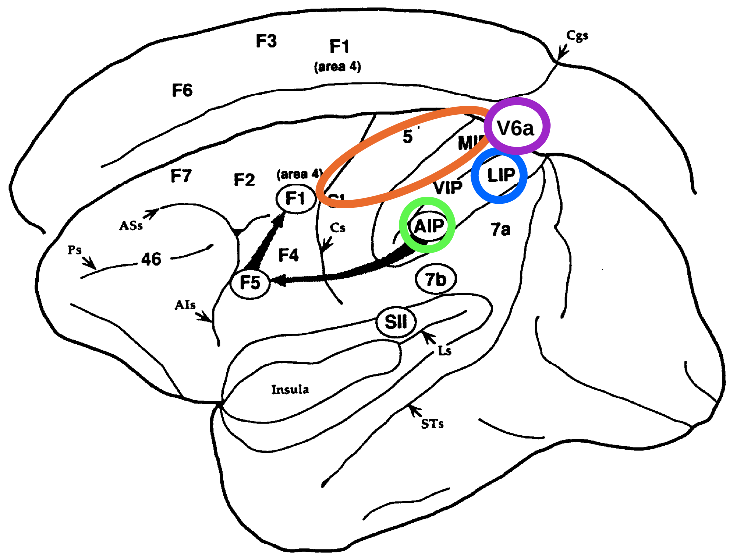
- The human capacity for arithmetic is founded upon an evolutionarily ancient representation of the approximate number of concrete sets (number sense), already present in other animals

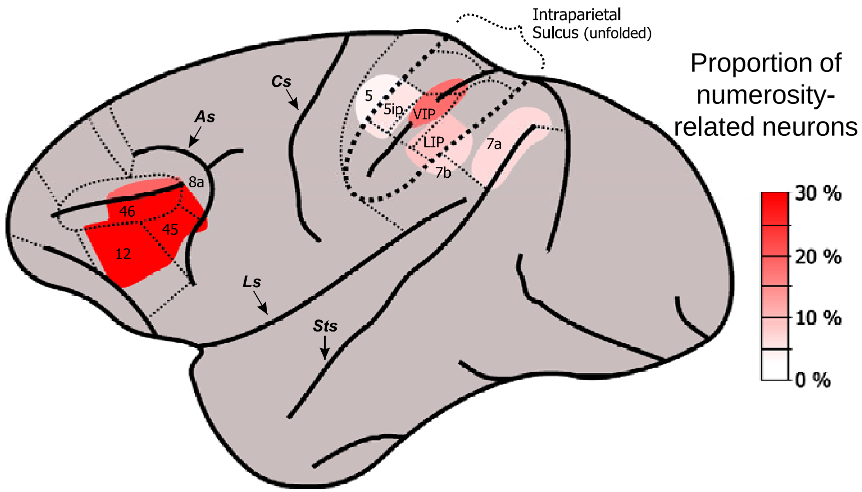
The intraparietal activation during calculation is surrounded by a geometrically reproducible array of sensori-motor areas

Human brain



Macaque monkey

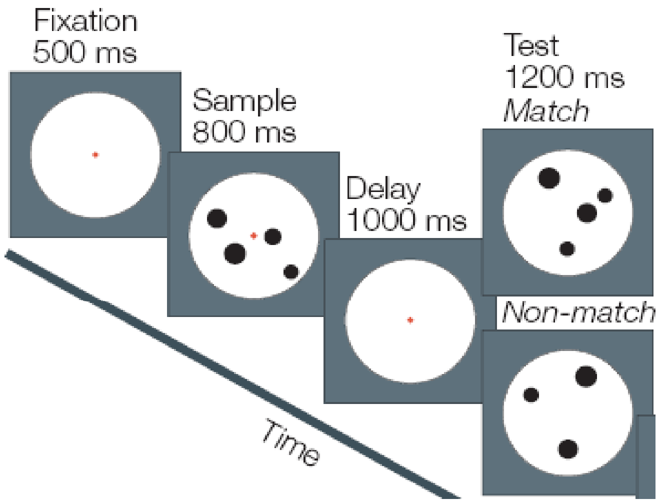




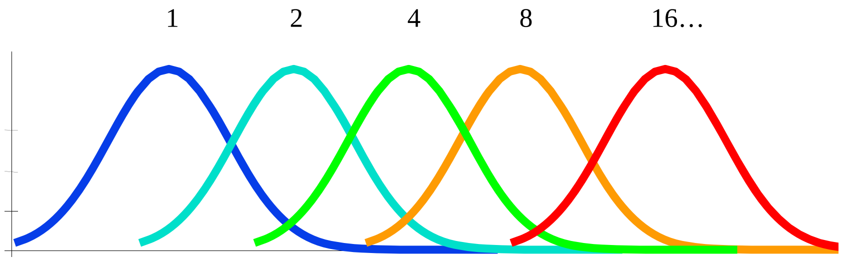
Number neurons in the monkey

(Nieder, Freedman & Miller, 2002; Nieder & Miller, 2003, 2004, 2005; Roitman, Brannon & Platt, 2007)

Task:
same-different judgement
with small numerosities



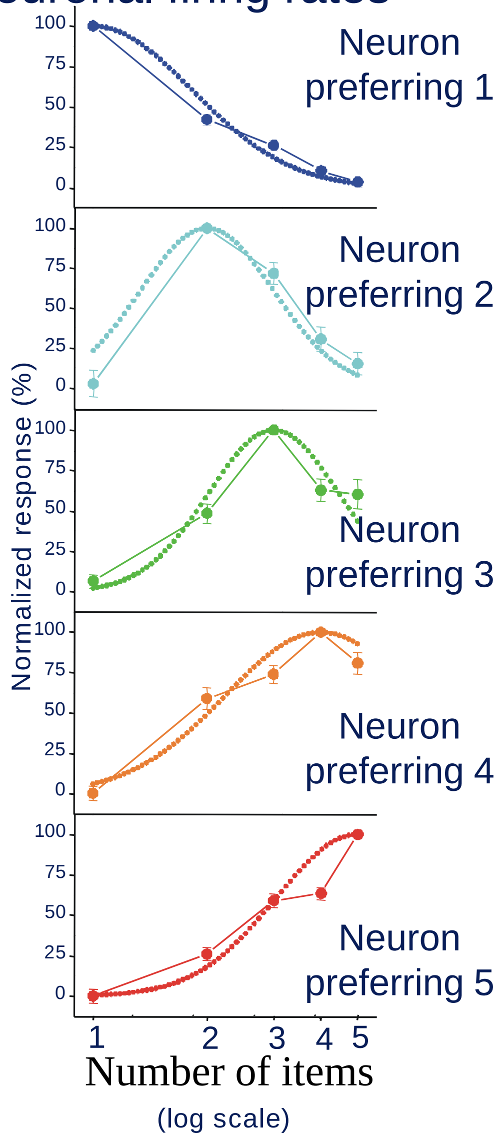
The Dehaene-Changeux (1993) model:
Coding by Log-Gaussian numerosity detectors



Internal logarithmic scale : log(*n*)

The concept of the « number line »

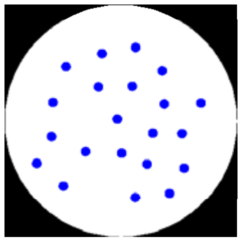
Neuronal firing rates



From numerosity detectors to numerical decisions: Elements of a mathematical theory

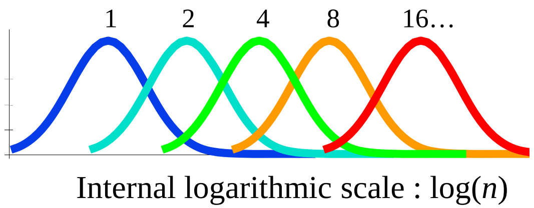
(S. Dehaene, Chapter in *Attention & Performance*, 2007, available from www.unicog.org)

Stimulus of numerosity *n*

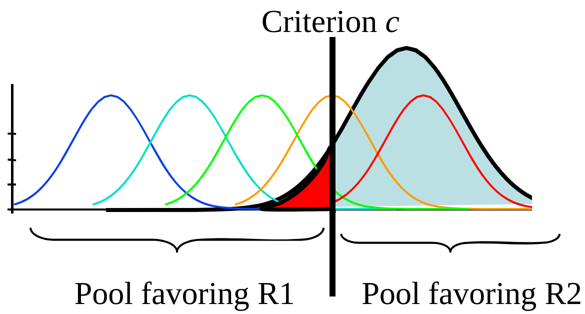


Response in simple arithmetic tasks:
-Larger or smaller than x?
-Equal to x?

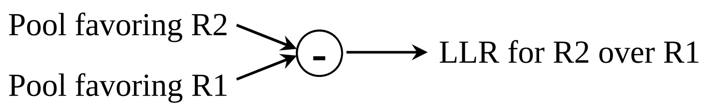
1. Coding by Log-Gaussian numerosity detectors



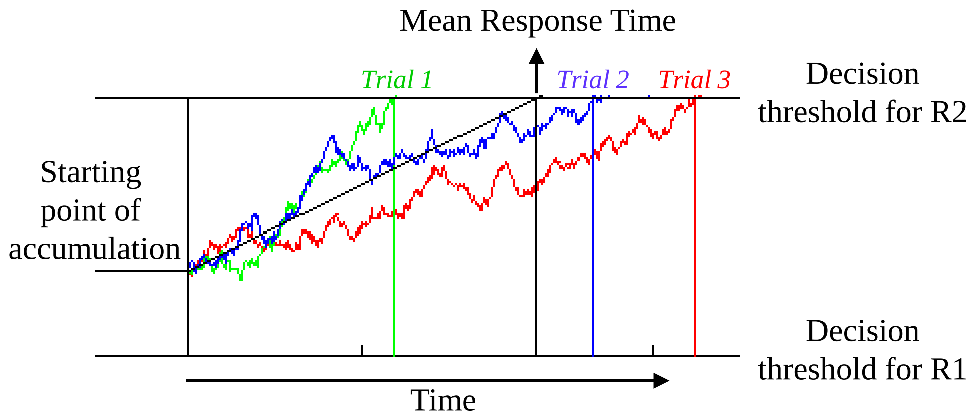
2. Application of a criterion and formation of two pools of units



3. Computation of log-likelihood ratio by differencing



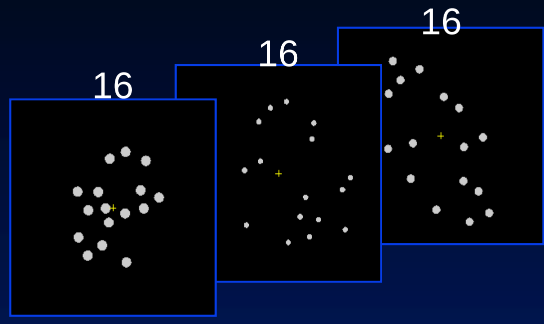
4. Accumulation of LLR, forming a random-walk process



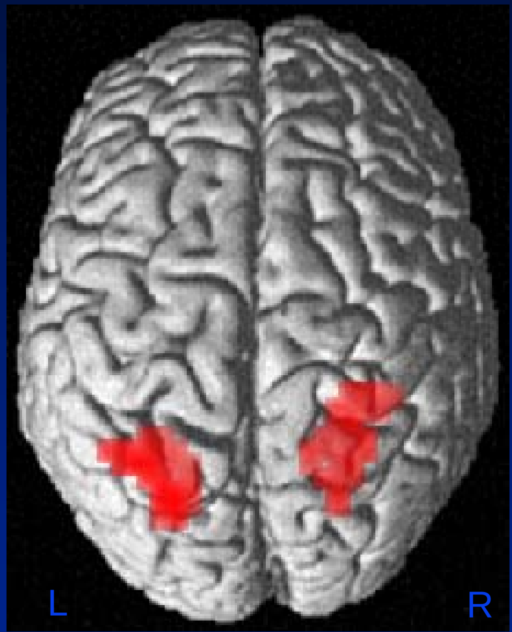
fMRI adaptation reveals Log-Gaussian turning in the human intraparietal sulcus

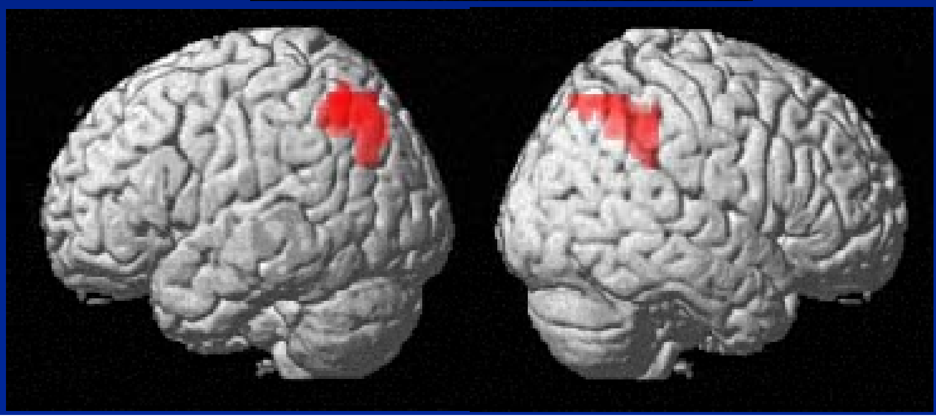
Piazza, Izard, Pinel, Le Bihan & Dehaene, Neuron 2004

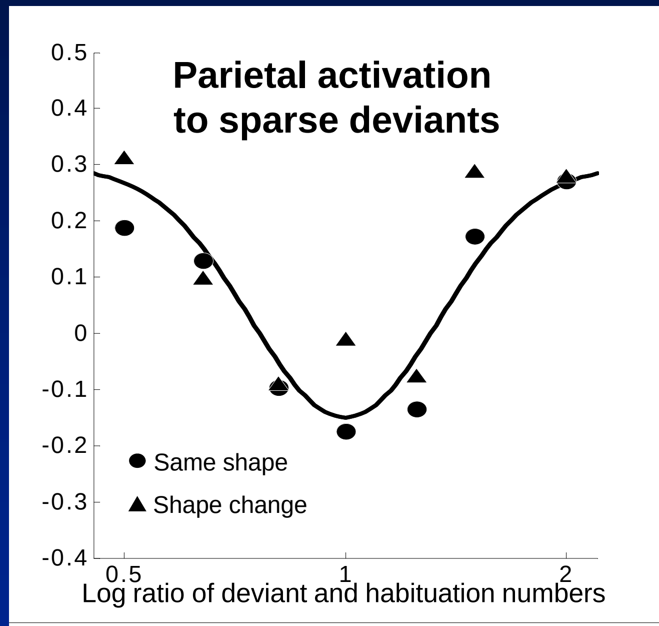
Adaptation to a fixed number

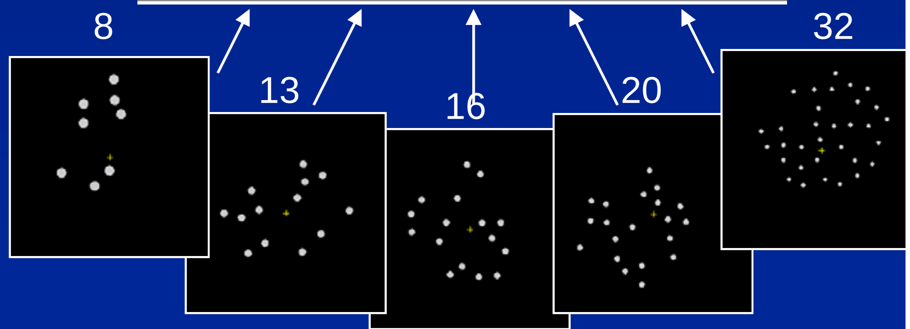


Regions responding to a change in number



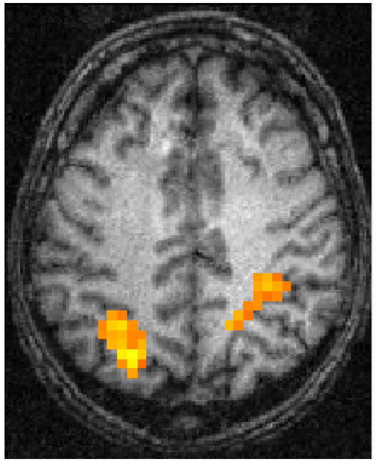






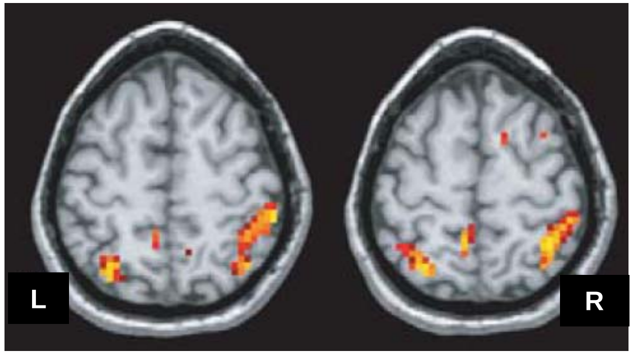
A basic dorsal-ventral organization for shape vs number

Initial study: effect of number change



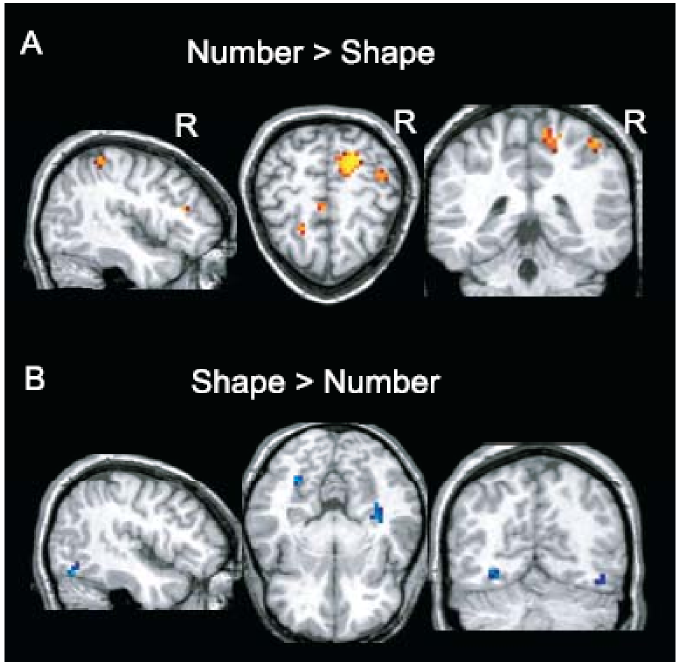
Number change in
intraparietal cortex

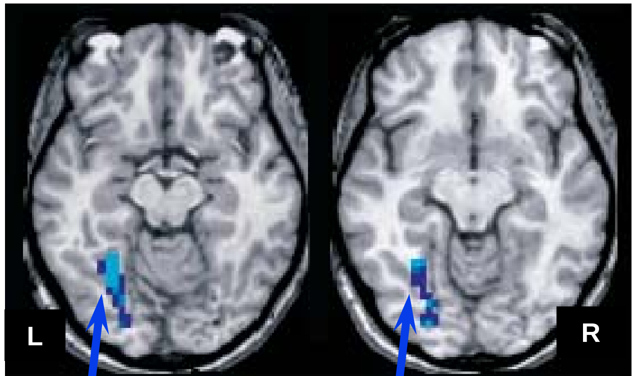
Improved design by Cantlon, Brannon et al. (PLOS, 2006):



Number change > Shape change in bilateral intraparietal sulci

Number and shape in four-year-olds





Shape change > Number change in left inferior temporal cortex

Could neuronal recycling extend to other domains of human competence?

Arithmetic intuitions in infants

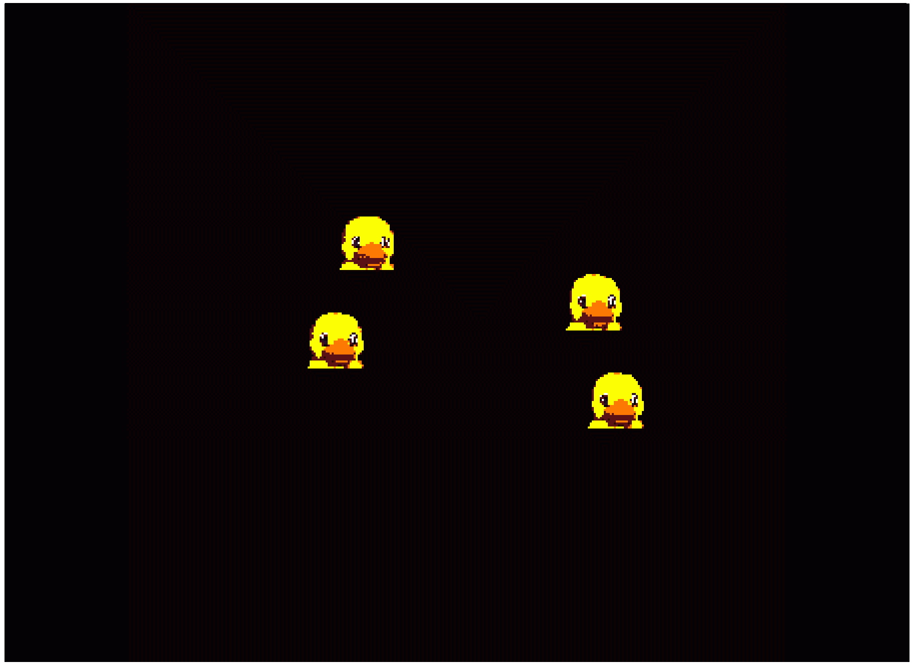
Babies of a few month of age discriminate numbers
and react to violations of the laws of arithmetic

When 5 +5 does not make 10….

…infants look longer at such impossible events

K. McCrink, K. Wynn

Numerosity adaptation in three month-old infants



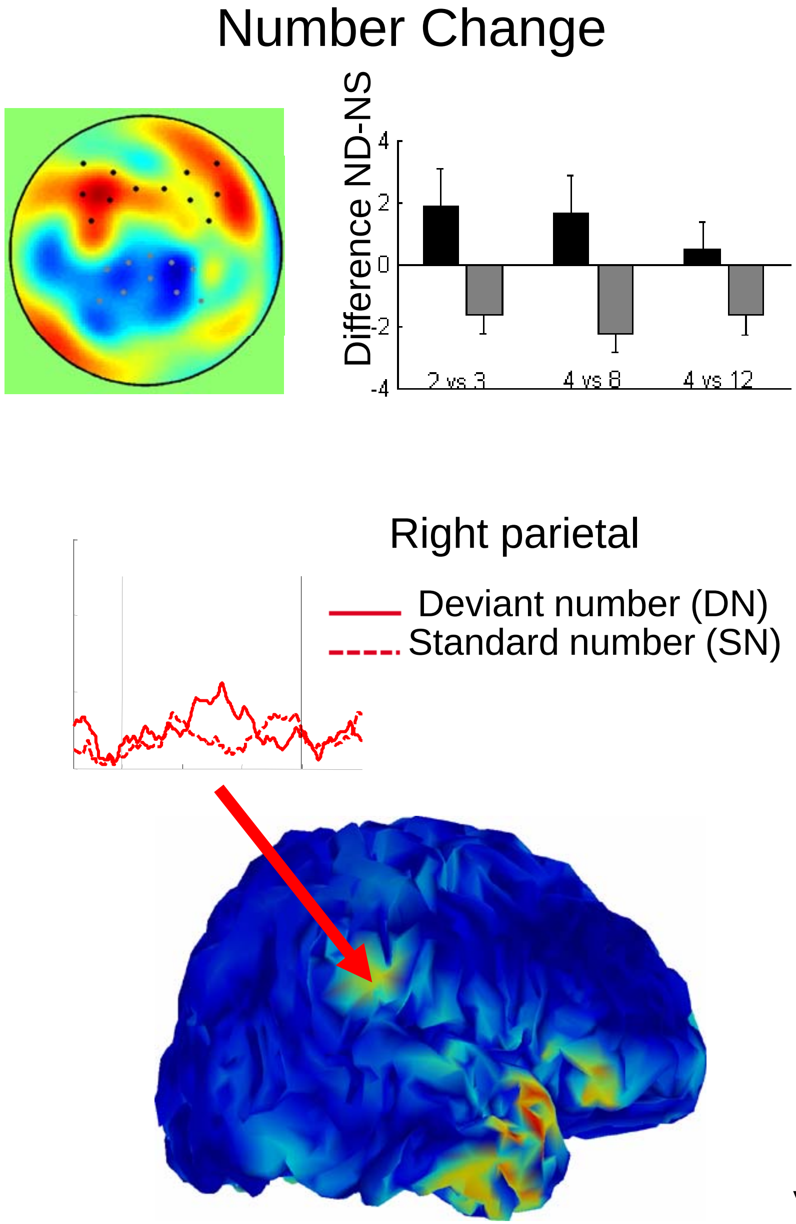
2 x 2 design : numerosity and/or object change

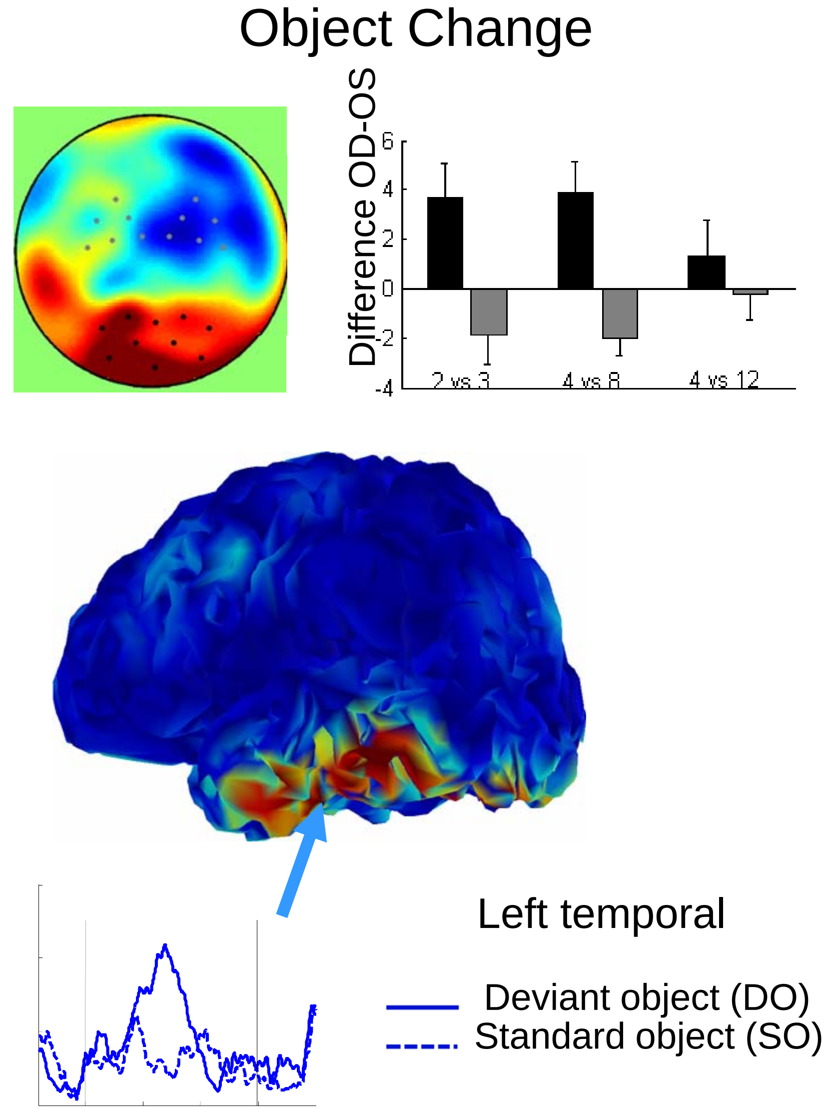
3 pairs of numerosities:
4 vs 8 ; 4 vs 12 ; 2 vs 3

Twelve 3-4 month-old infants in each group

Véronique Izard
Ghislaine Dehaene-Lambertz
PLOS Biology, 2008

Distinct dorsal and ventral circuits for number and object change in 3-month-old infants

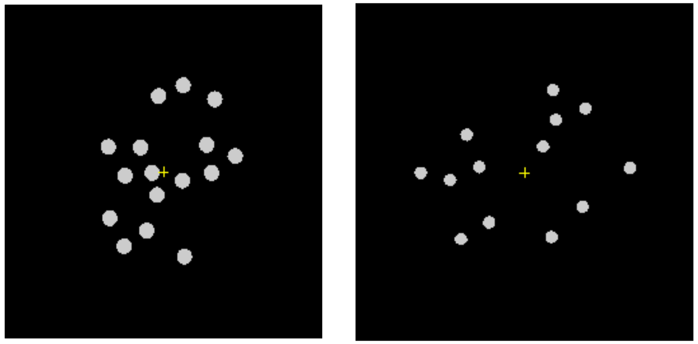




Véronique Izard, Ghislaine Dehaene-Lambertz, PLOS Biology, 2008

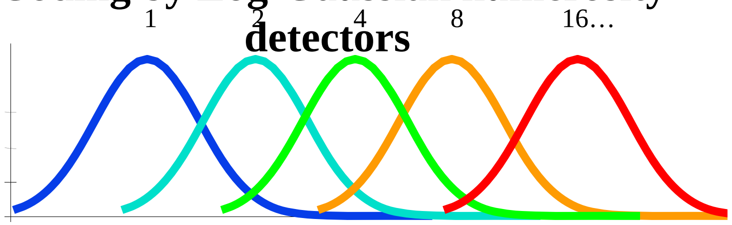
Development of precision in numerosity coding

(Piazza, Zorzi, Dehaene et al., submitted)

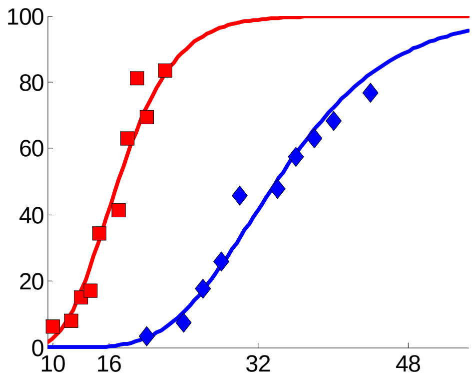


What is the larger number?

The Dehaene-Changeux (1993) model: Coding by Log-Gaussian numerosity detectors

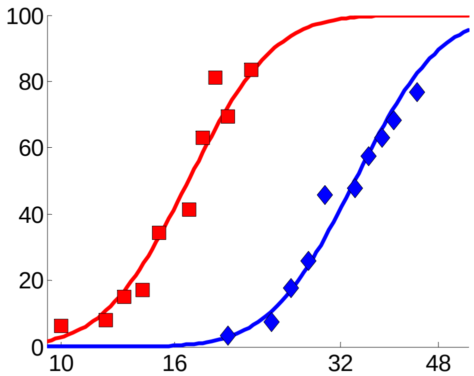


Internal logarithmic scale : log(*n*)
A single free parameter:
w, the precision of the code

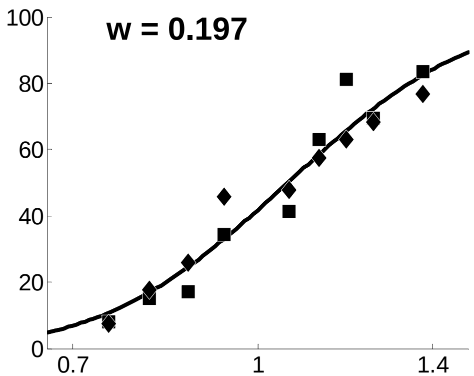


% « larger » responses

Linear scale



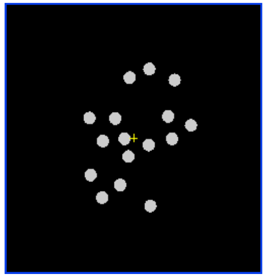
Log scale

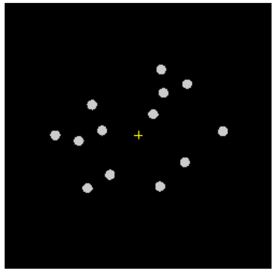


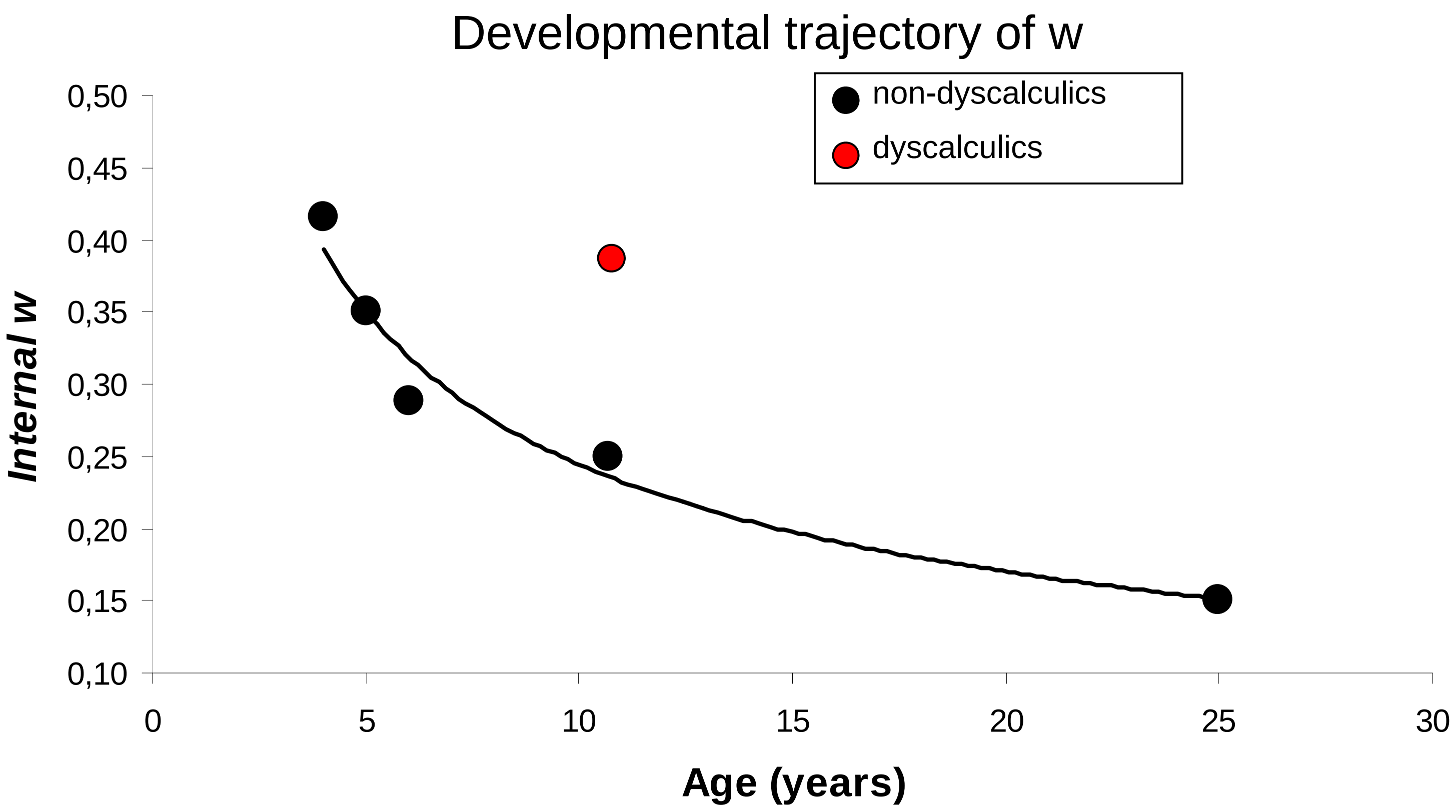
Log ratio scale

Development of precision in numerosity coding

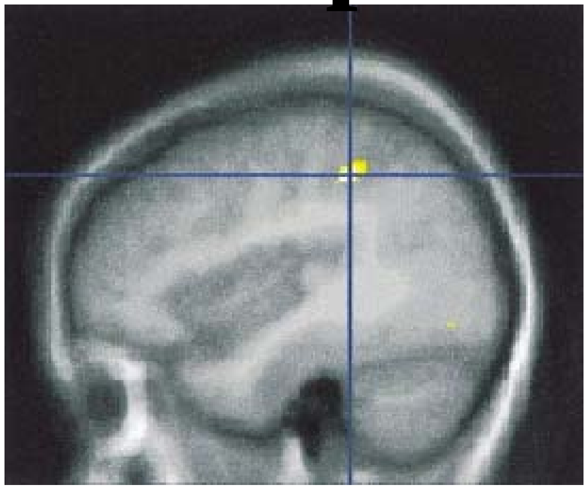
(Piazza, Zorzi, Dehaene et al., submitted)

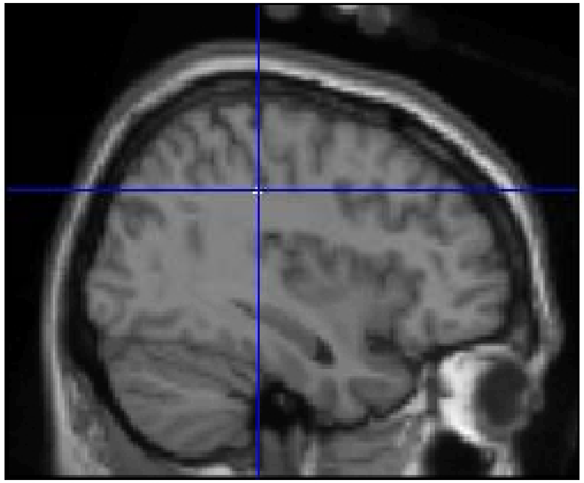


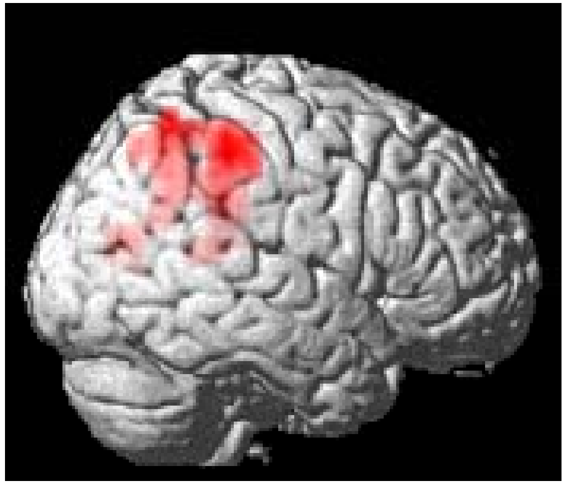




Developmental dyscalculia:
An impairment in number sense?

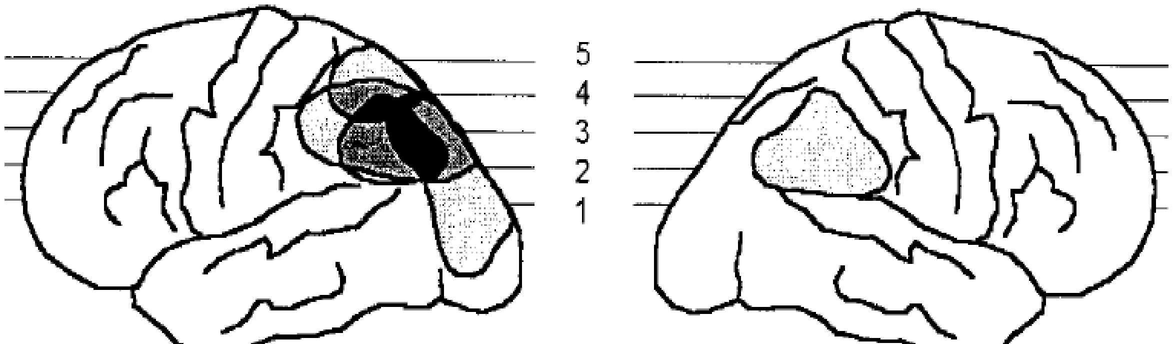






Dyscalculic adults born pre-term show missing gray matter in the intraparietal sulcus, compared to non-dyscalculic pre-term controls. (Isaacs et al., 2001)

Turner's syndrome (monosomy 45-X) is frequently associated with dyscalculia. We found that a group of Turners girls showed both structural and functional alterations in the intraparietal sulcus (Molko et al., 2003)



Lesions causing acalculia in adults

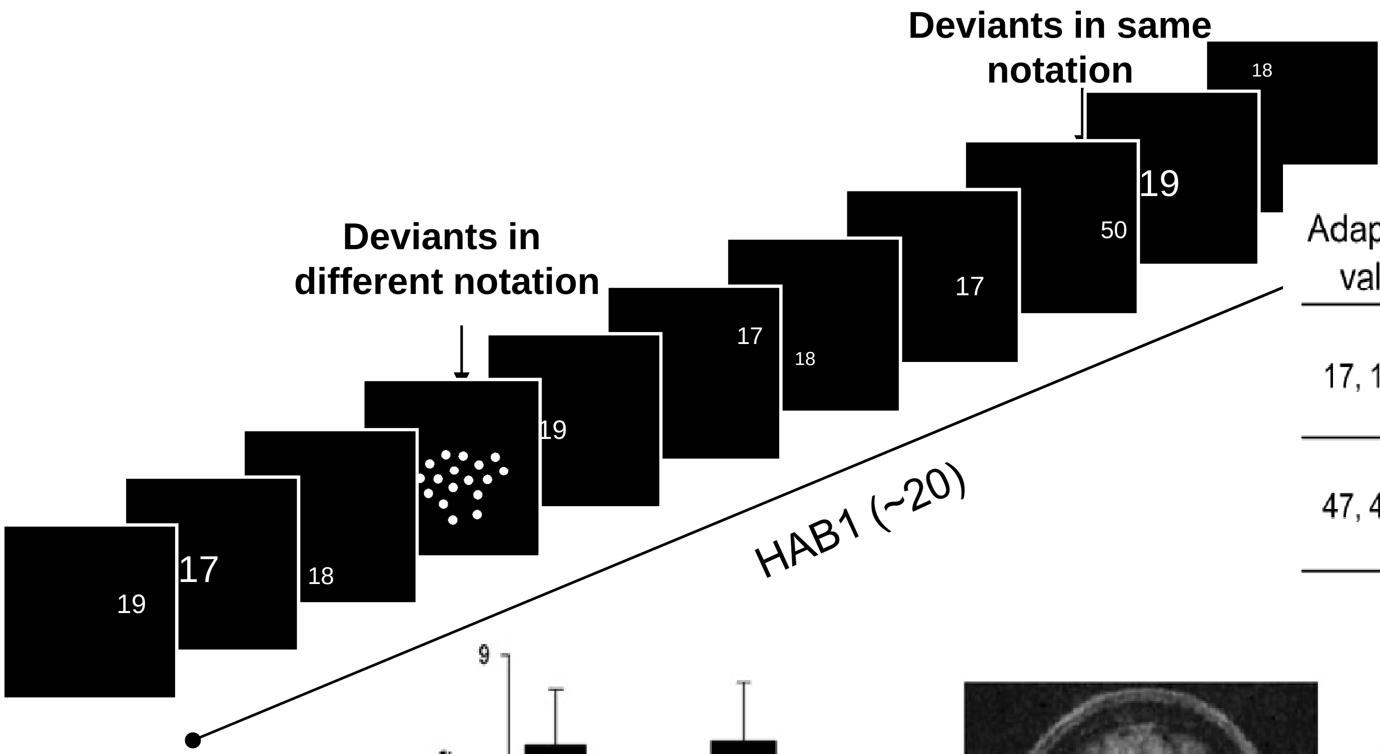
Hypothesis 2

- Education in humans makes the approximate number representation available from words and symbols

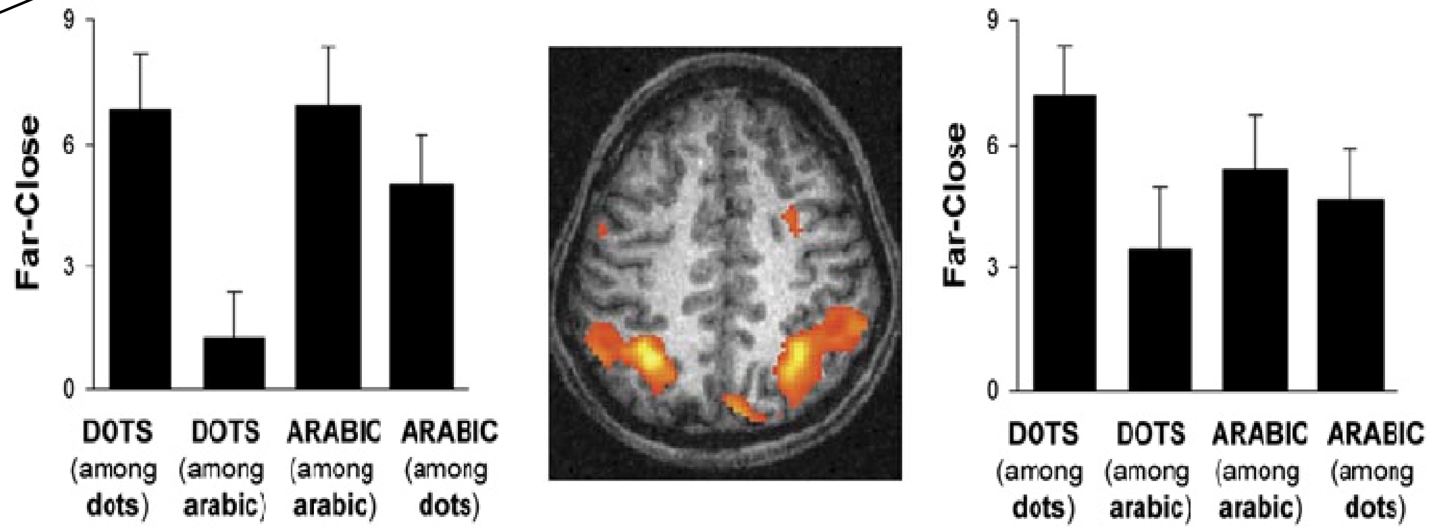
An fMRI study of cross-notation adaptation

Piazza, Pinel and Dehaene, Neuron 2007

- Do the same neurons code for the symbol 20 and for twenty dots?



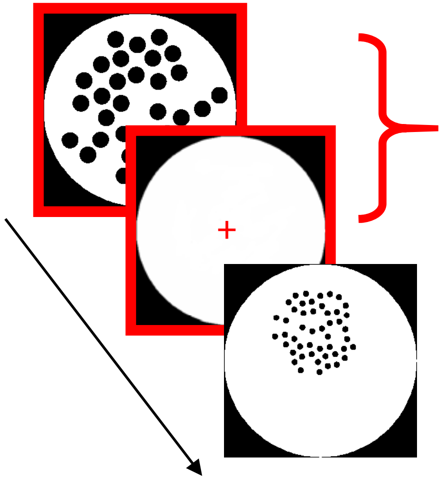
Adaptation values	Deviant values 20	50
17, 18, or 19	close	far
47, 48, or 49	far	close

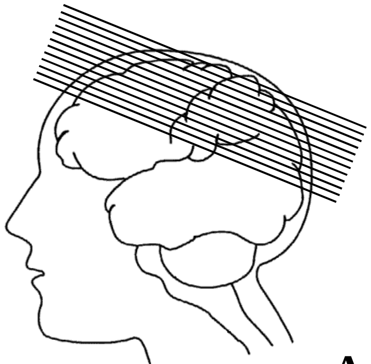


Decoding numerosity from intraparietal fMRI signals

Eger, Michel, Thirion, Amadon, Dehaene & Kleinschmidt, submitted

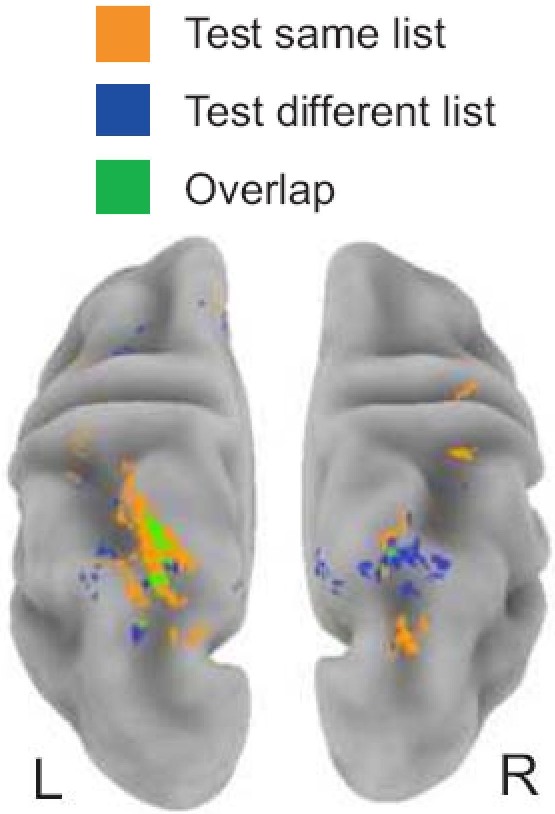
Numerosities 4, 8, 16, 32
Memorize the numerosity
and match it to a second set

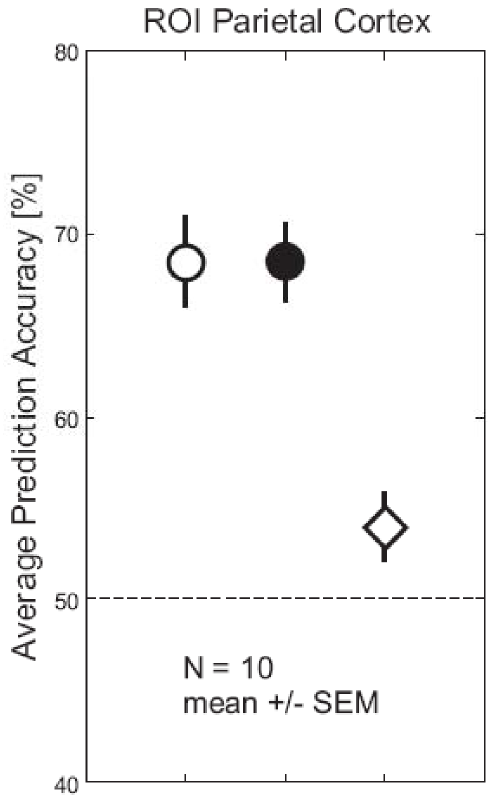




high-res fMRI
1.5 mm voxels
at 3T

A multivariate classifier can infer number from the pattern of brain activity, and generalize across two stimulus sets controlled for non-numerical variables



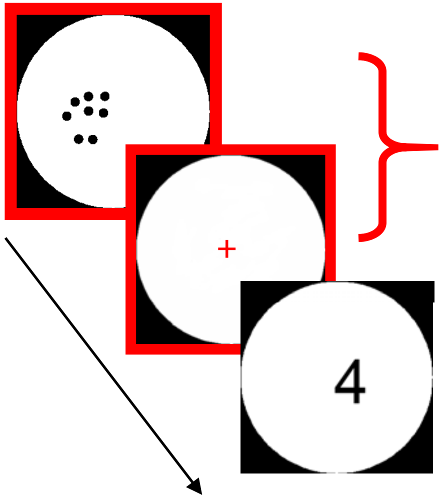


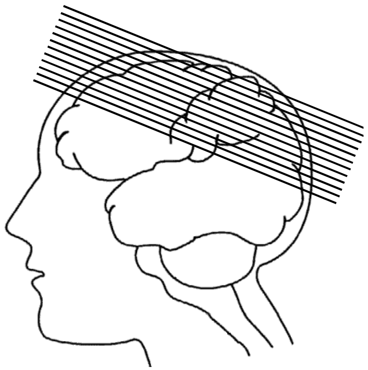
A « searchlight » approach shows that the most informative voxels are in the IPS

Decoding numerosity from intraparietal fMRI signals

Eger, Michel, Thirion, Amadon, Dehaene & Kleinschmidt, submitted

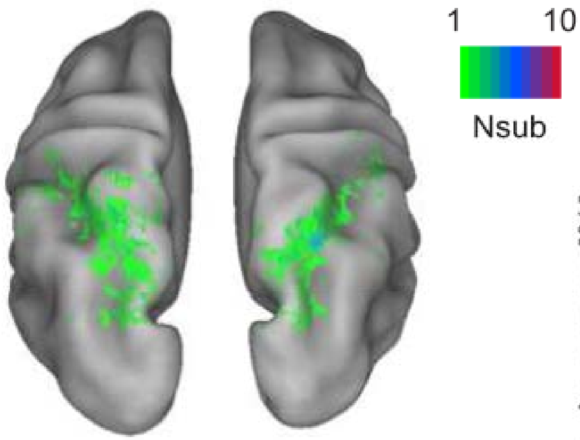
Numerosities 2, 4, 6, 8
Samples and targets can be presented in Dot or Arabic notation

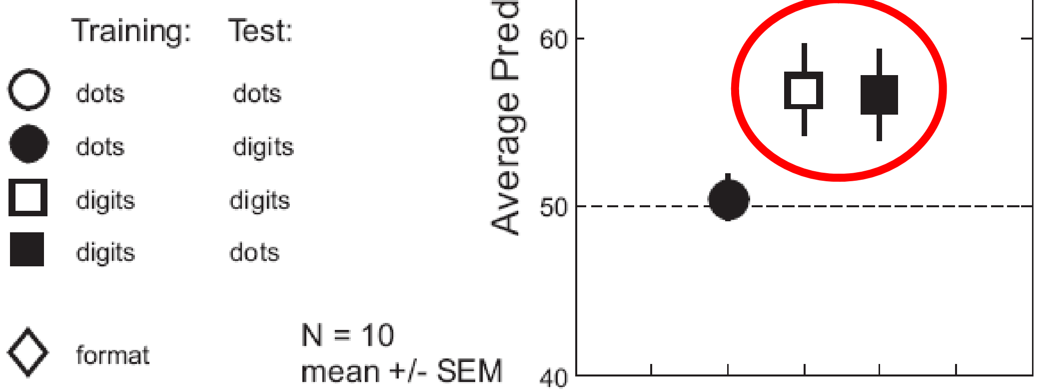




high-res fMRI
1.5 mm voxels
at 3T

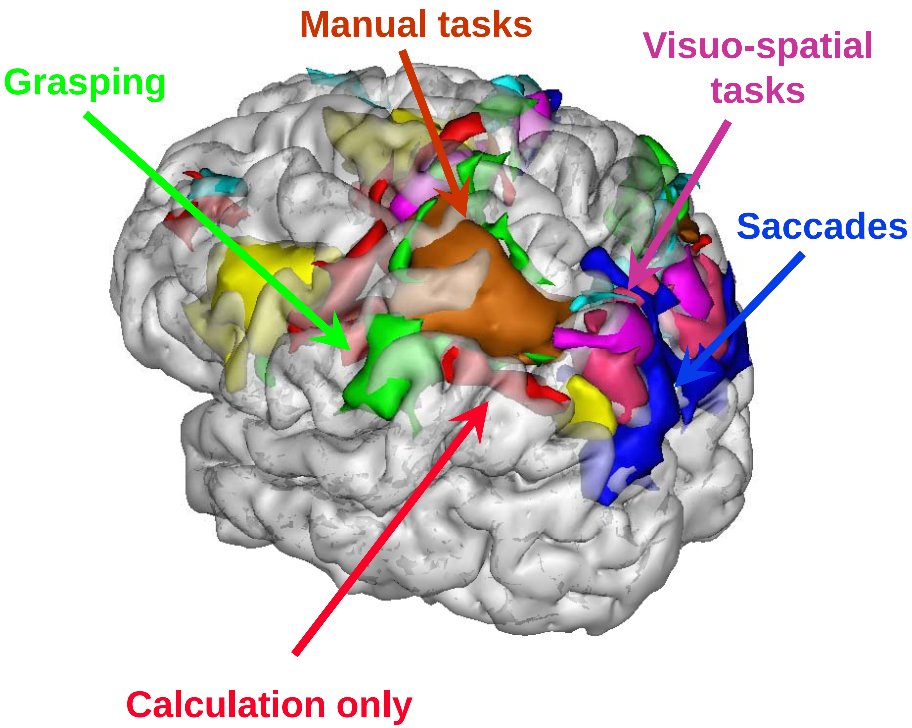
A multivariate classifier, trained with Arabic digits, generalizes equally well to sets of dots!





- **Hypothesis 3: Arithmetic recycles nearby areas involved in visuo-spatial transformations**

Human brain



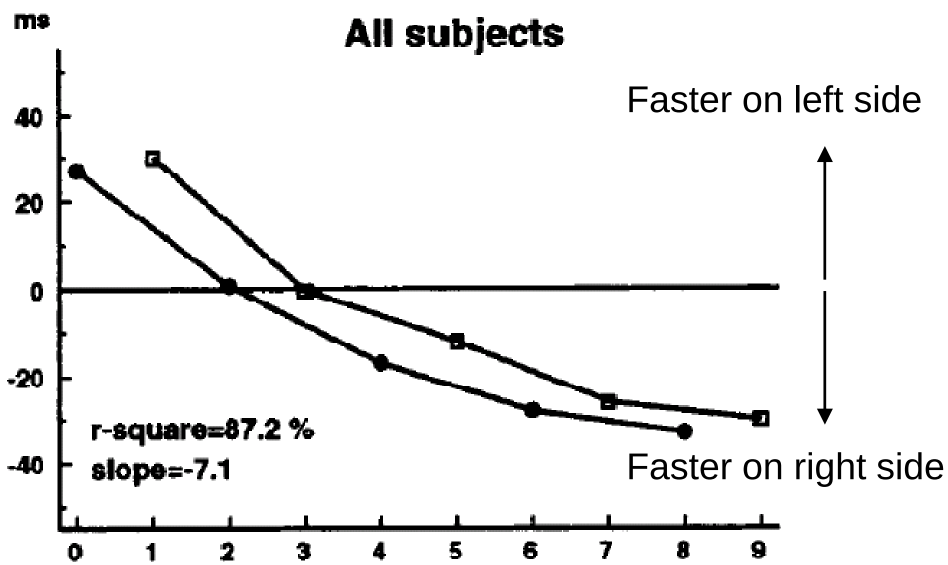
Interactions between Number and Space

Spatial-**N**umerical **A**ssociation
of **R**esponse **C**odes

= **SNARC effect**

(Dehaene et al., 1993)

RT(right key) minus RT(left key)

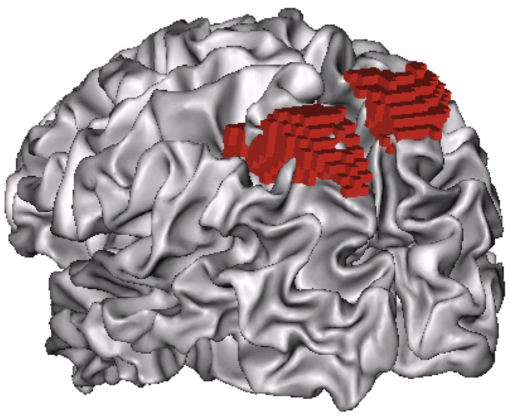


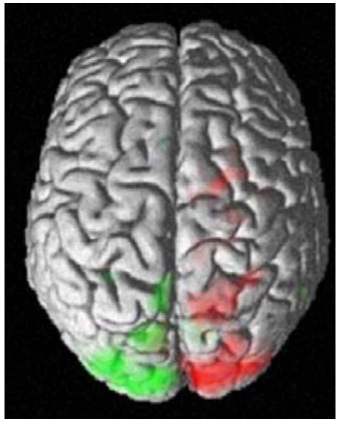
Hemispatial neglect
in numerical bisection task
(Zorzi et al., 2002)

Cross-talk between number and space during calculation

Knops, Thirion, Hubbard & Dehaene, *Science*, 2009

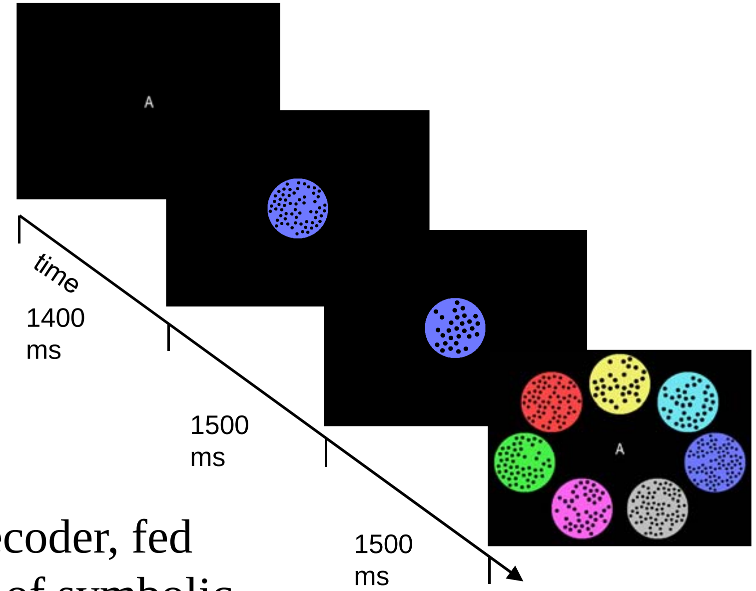
Training block: eye movements





- Decoding eye movements to the left (red) vs. to the right (green)
- The decoder predicts novel left or right trials with 70% accuracy on average (range: 56%-85%). Classification is above chance ($p<0.05$) in 14/15 subjects.

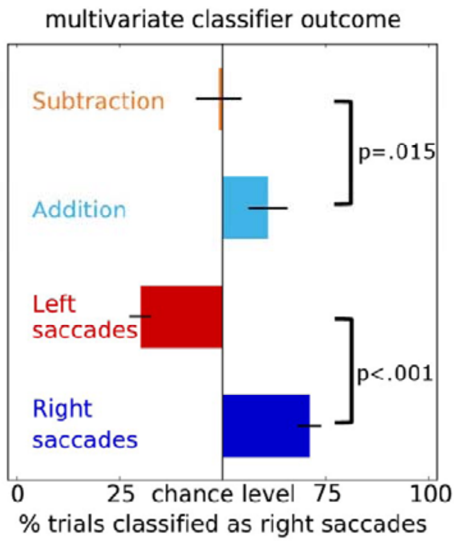
Test block: Addition / Subtraction



- The same decoder, fed with images of symbolic or non-symbolic calculation, generalizes:

The distinction between left and right eye movements can also be used to distinguish subtraction from addition

(with Arabic or Dot notation)



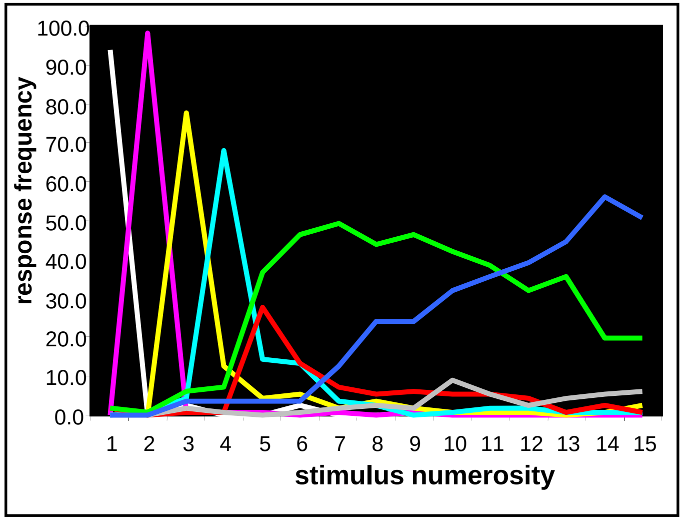
Changes due to education: Studies in the Munduruku

with Véronique Izard, Pierre Pica, and Liz Spelke; *Science*, 2004, 2006, 2008
Brasilian collaborators/consultants: André Ramos (Funai), Ana Arnor, Venancio Poxõ, André Ramos, Celso Tawé, Felix Tawé, André Tawé, Miguel Karu

- pug ma = one
- xep xep = two
- ebapug = three
- ebadipdip = four
- pug põgbi = one hand
- xep xep põgbi = two hands
- adesu/ade gu = some, not many
- ade/ade ma = many, really many

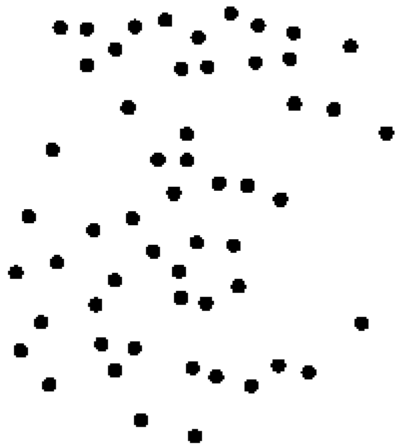
A reduced lexicon of number words

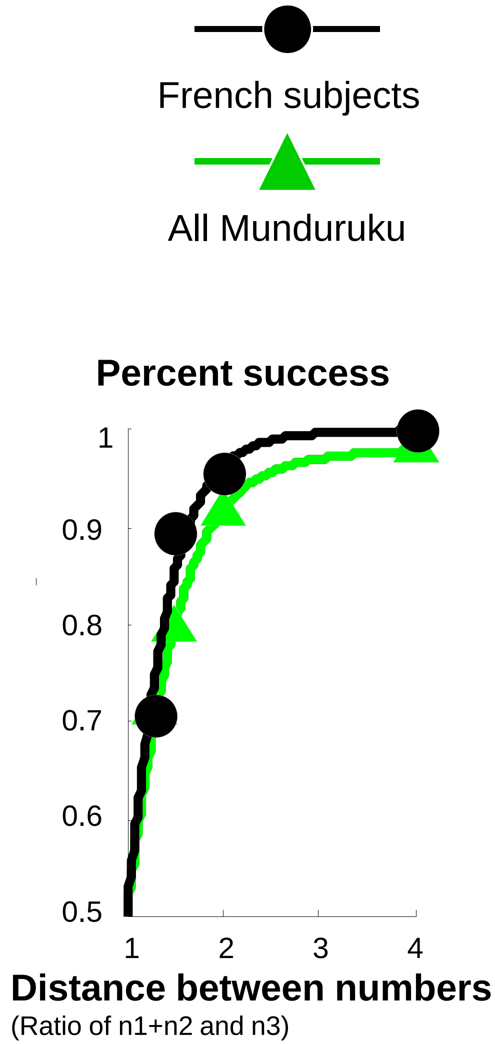
Munduruku number words refer to approximate numerosity



Munduruku adults and children can perform approximate arithmetic with non-verbal numerosities (e.g. 40+30 is larger than 50) but not exact arithmetic (e.g. 7-6=1)

Success in approximate addition and comparison

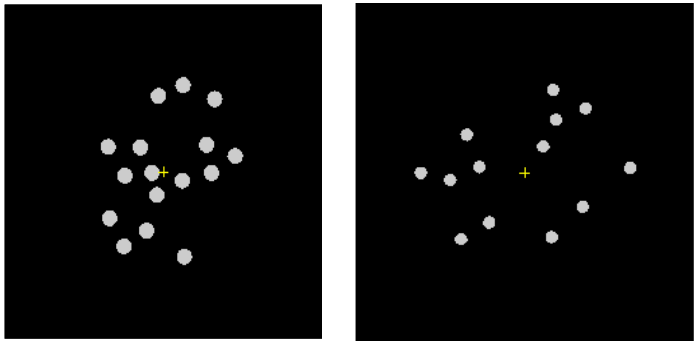


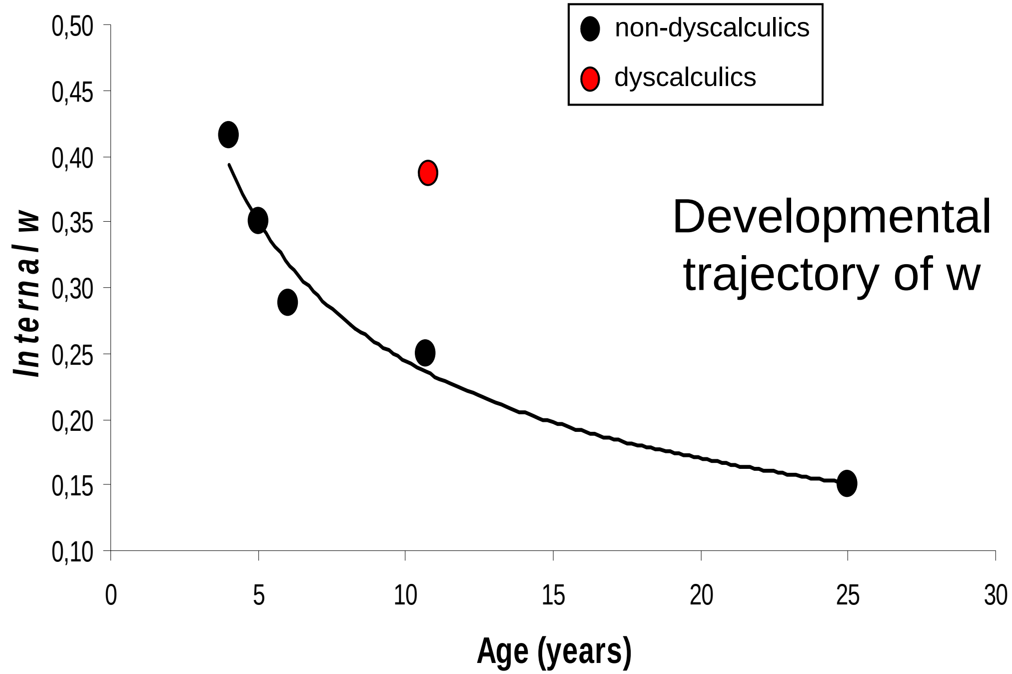


Pica, Lemer, Izard, & Dehaene, Science, 2004

Development of precision in numerosity coding

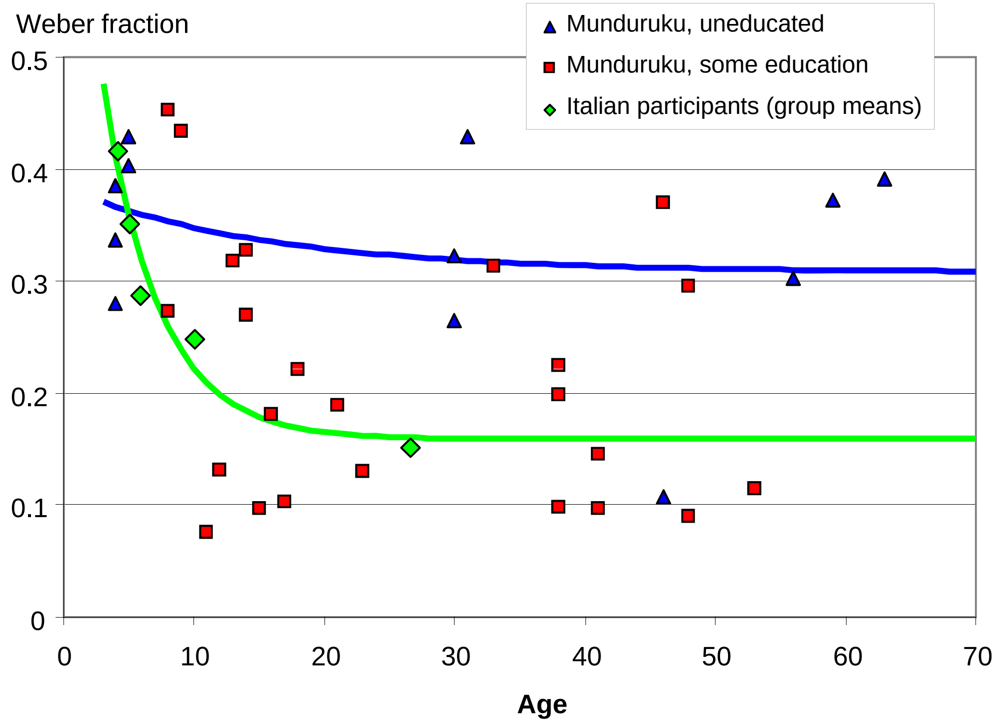
(Piazza, Zorzi, Dehaene et al., submitted)

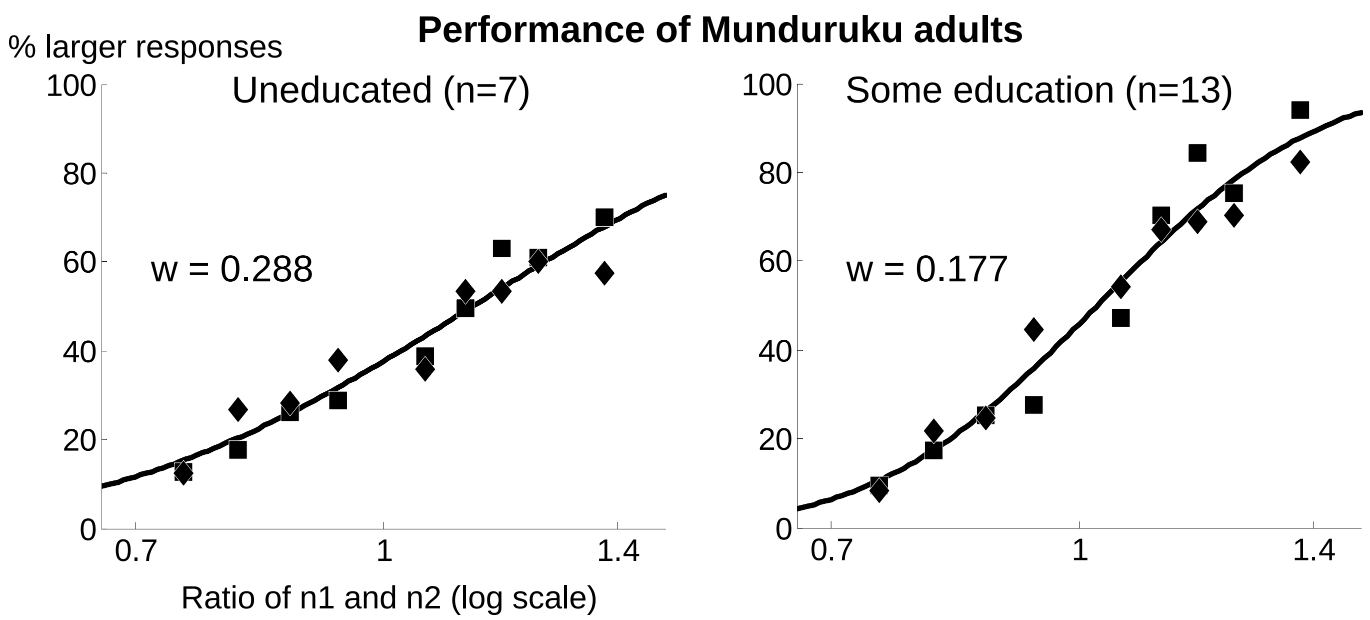


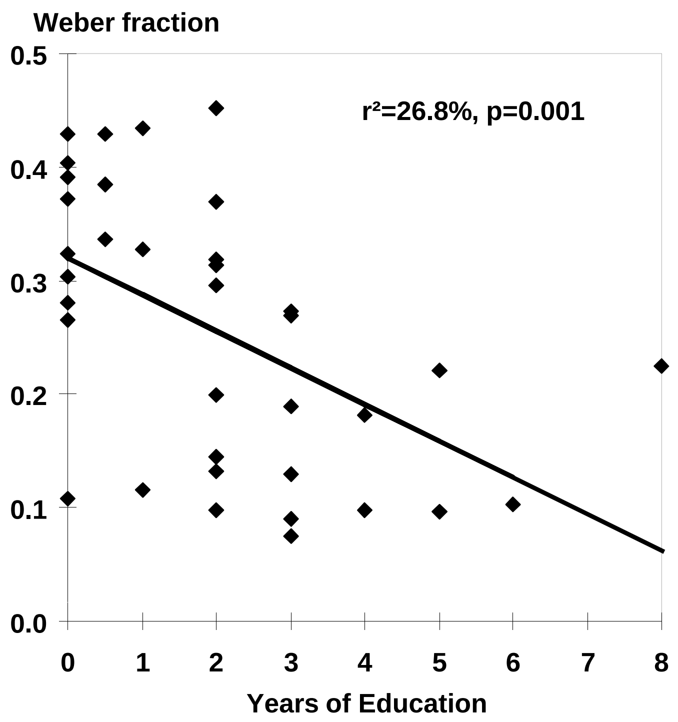


In the Munduruku, the precision of the approximate number system depends on education

(Dehaene, Pica, Piazza et al, in prep.)





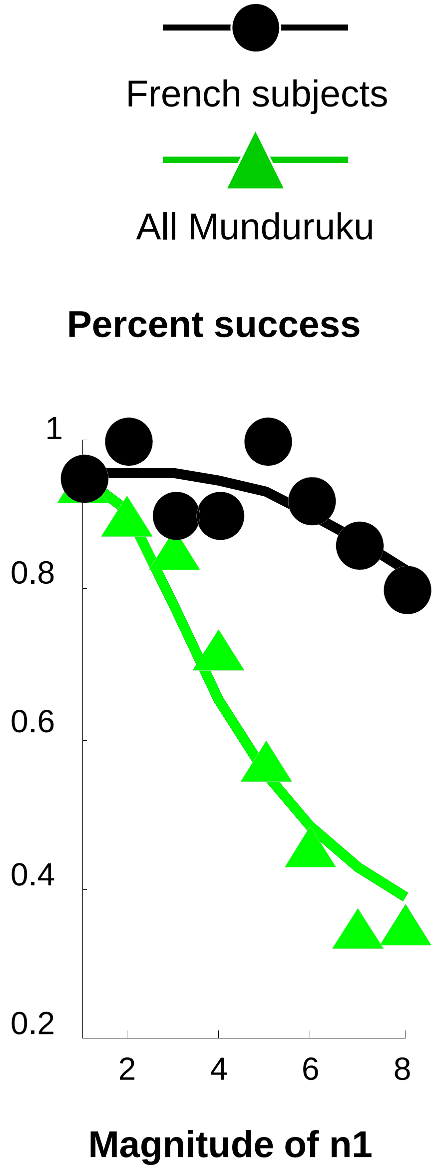


Conclusion:

The approximate number system is universal and present early on.

Both maturation and education contribute to its progressive refinement.

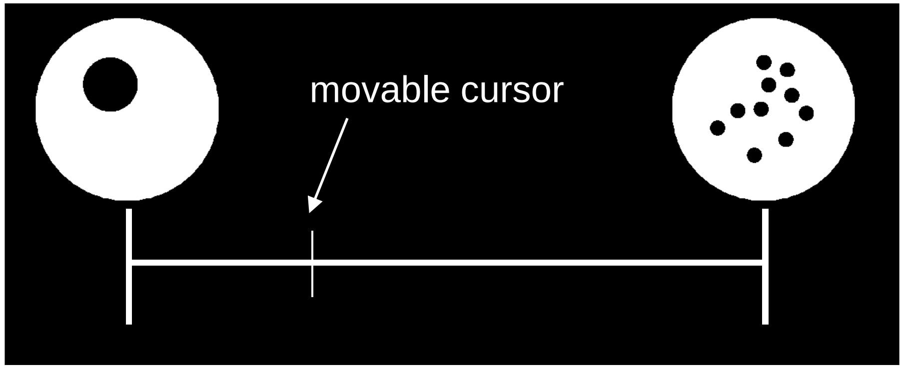
Failure in exact subtraction of small quantities

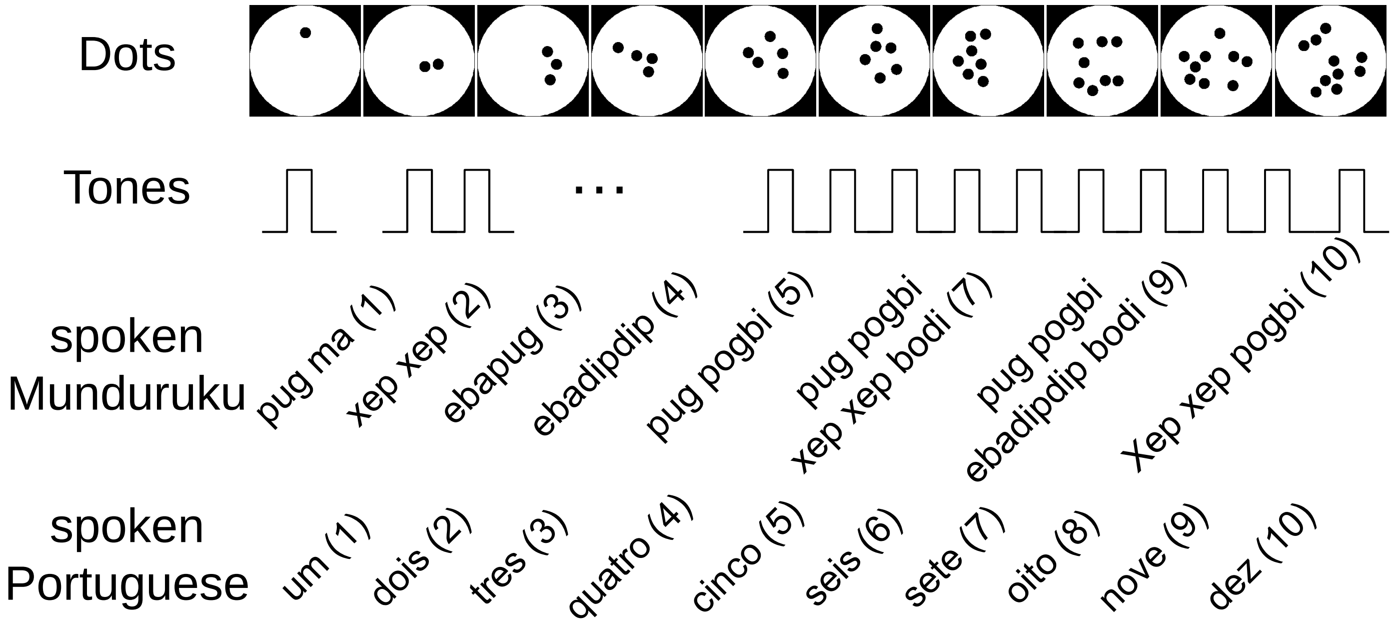


Pica, Lemer, Izard, & Dehaene, Science, 2004

Number-Space mapping in the Munduruku

Munduruku children and adults were asked to point to the location corresponding to a certain number. Would they show a compressive mapping even in adults? And for numbers as small as 1-10?





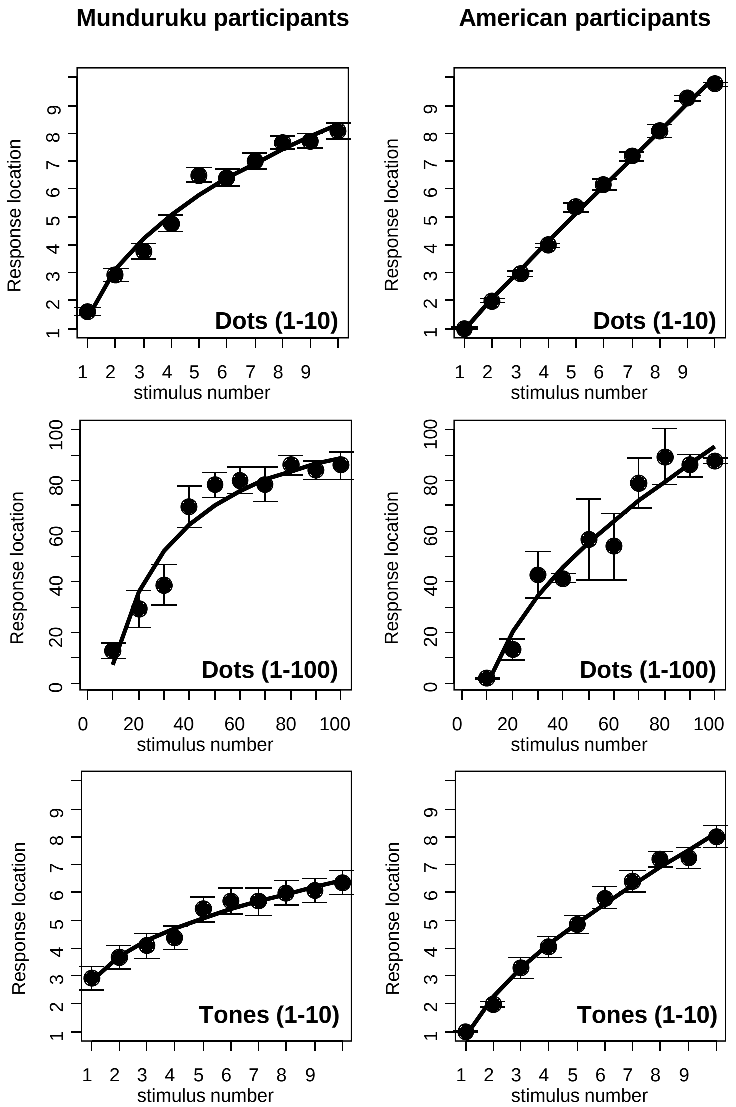
Dehaene, Izard, Spelke and Pica, Science 2008

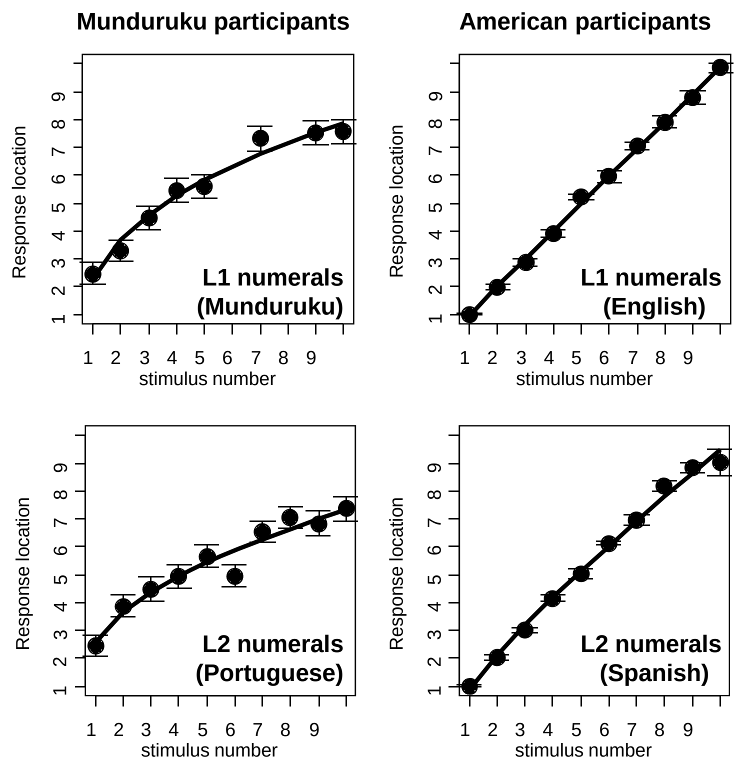
Logarithmic Number-Space mapping in the Munduruku

Munduruku children and adults show a compressive mapping

- For dot patterns or series of 1-10 tones

- For Munduruku words and even for Portuguese numerals





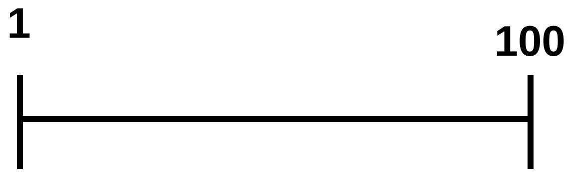
Dehaene, Izard, Spelke and Pica, Science 2008

Development of the linear understanding of number

(Siegler & Opfer, 2003; Siegler & Booth, 2004)

Number-Space mapping task:

« Please point to where number x should fall »



A major change occurs during mathematical education : switch from a logarithmic to a linear understanding of number

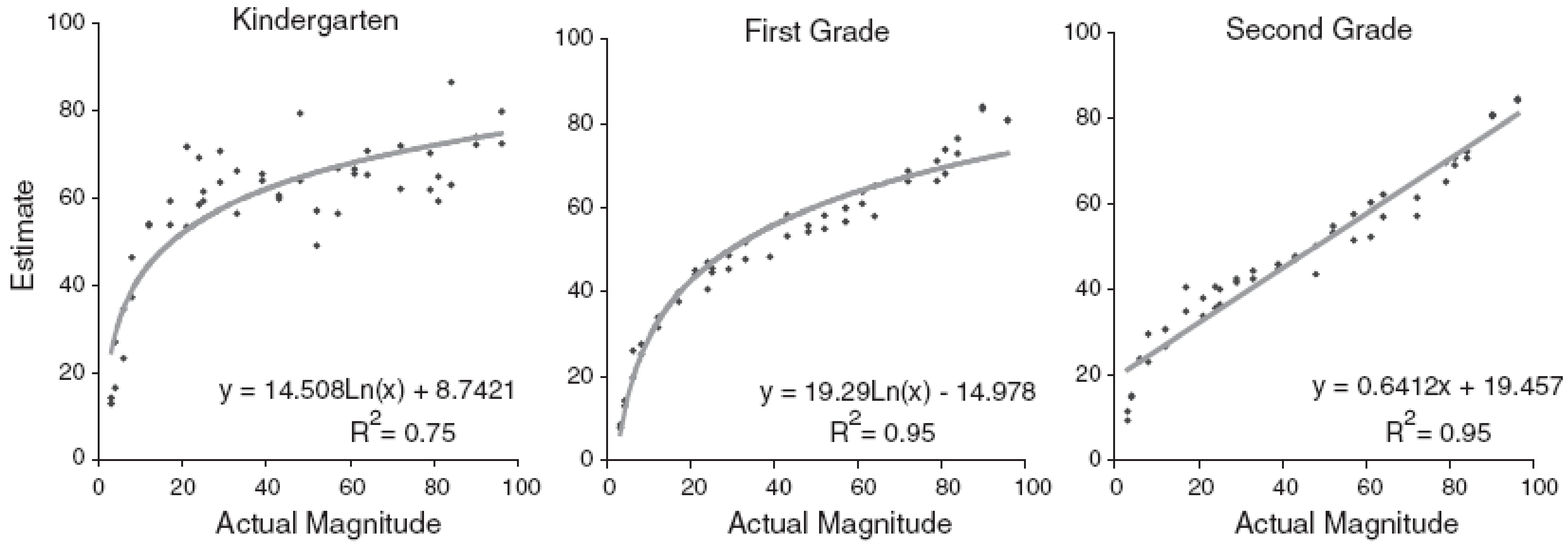
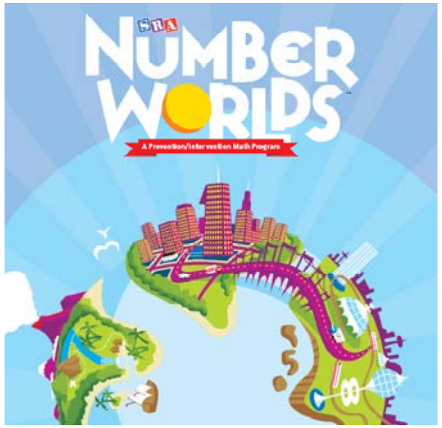


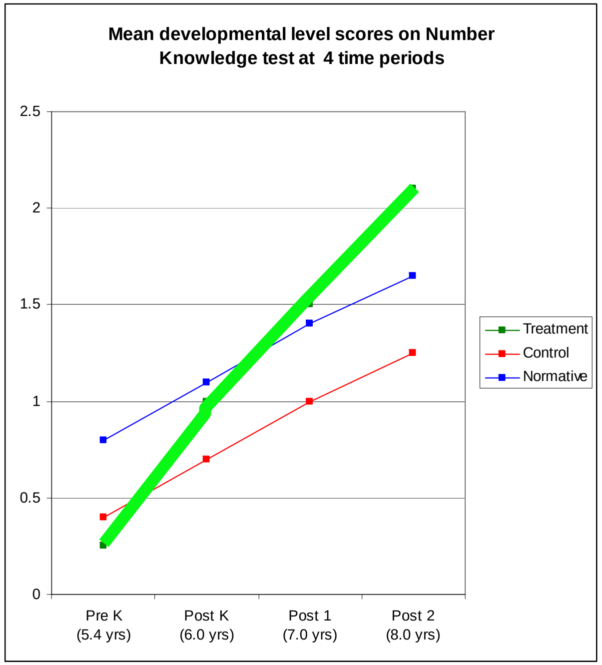
Figure 2. Progression from logarithmic pattern of median estimates among kindergartners (left panel) to linear pattern of estimates among second graders (right panel) in Experiment.

Playing board games promotes arithmetic in low-SES kindergartners

Griffin & Case (1994, 2004)

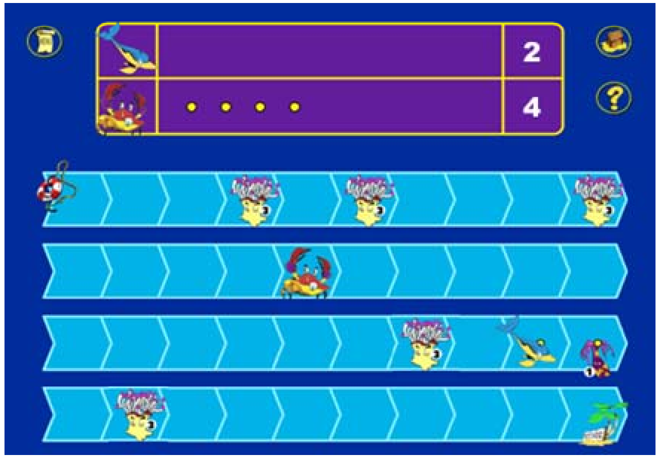
Training with a curriculum involving board games leads to long-lasting improvements



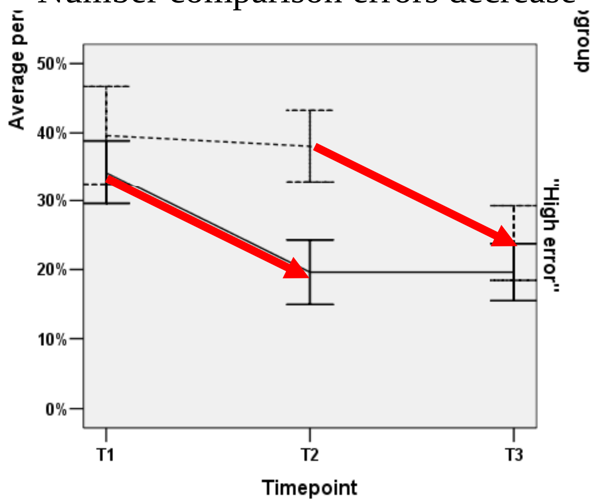


Wilson & Dehaene (2007), with Fayol

NumberRace Software improves subitizing, comparison, identification (relative to reading software)

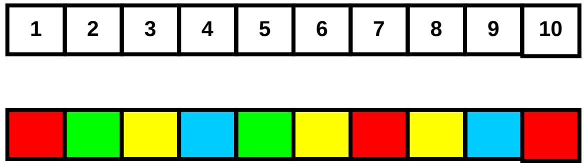


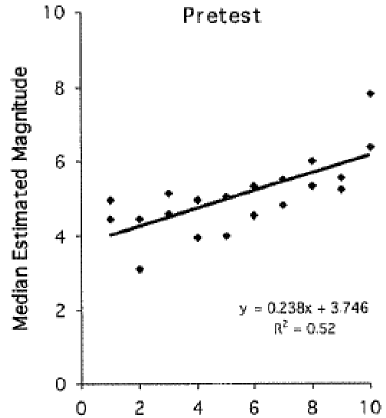
Number comparison errors decrease

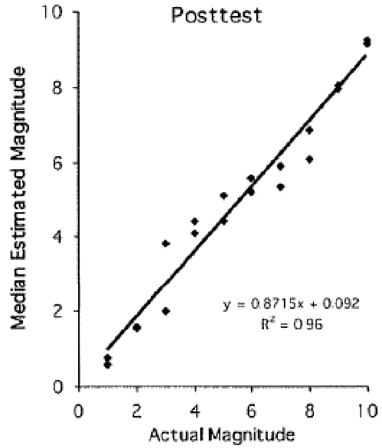


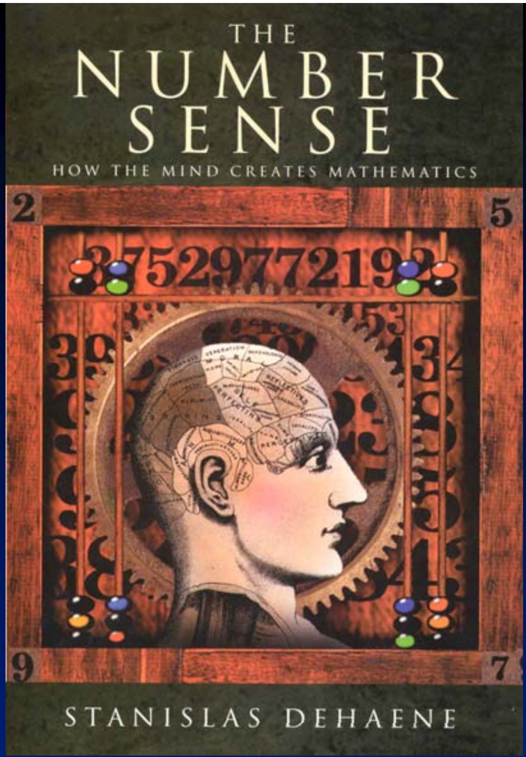
Ramani & Siegler (2008)

Very brief training leads to improvements in number-space mapping, comparison, identification, counting (relative to control game)









The number sense hypothesis and the foundation of mathematics

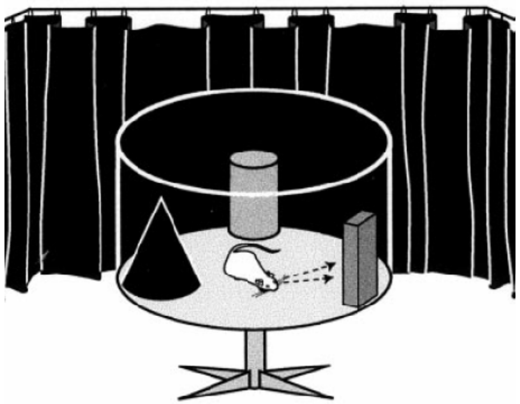
- We are all born with a « number sense », a specific ability to represent the approximate number of objects in concrete sets, and to combine these numbers into simple operations
- Acquisition of number symbols is founded on this pre-existing number sense: we learn to connect symbols to approximate quantities.
- Arithmetic « recycles » parietal lobe circuits for numerosity perception and for spatial transformations (neuronal recycling model)

- We are now extending this research to geometrical concepts

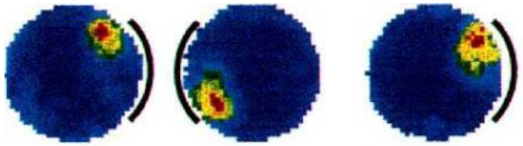
Core knowledge of geometry

Is **geometry**
also part of our evolutionary heritage,
much like number sense is?

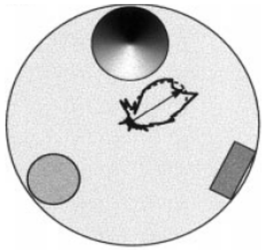
Animal navigation abilities



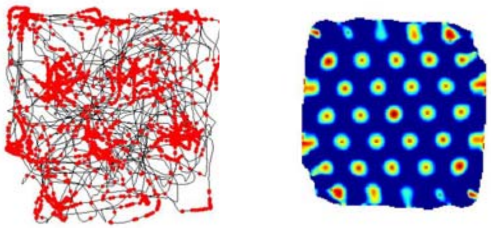
Place cells



Head direction cells



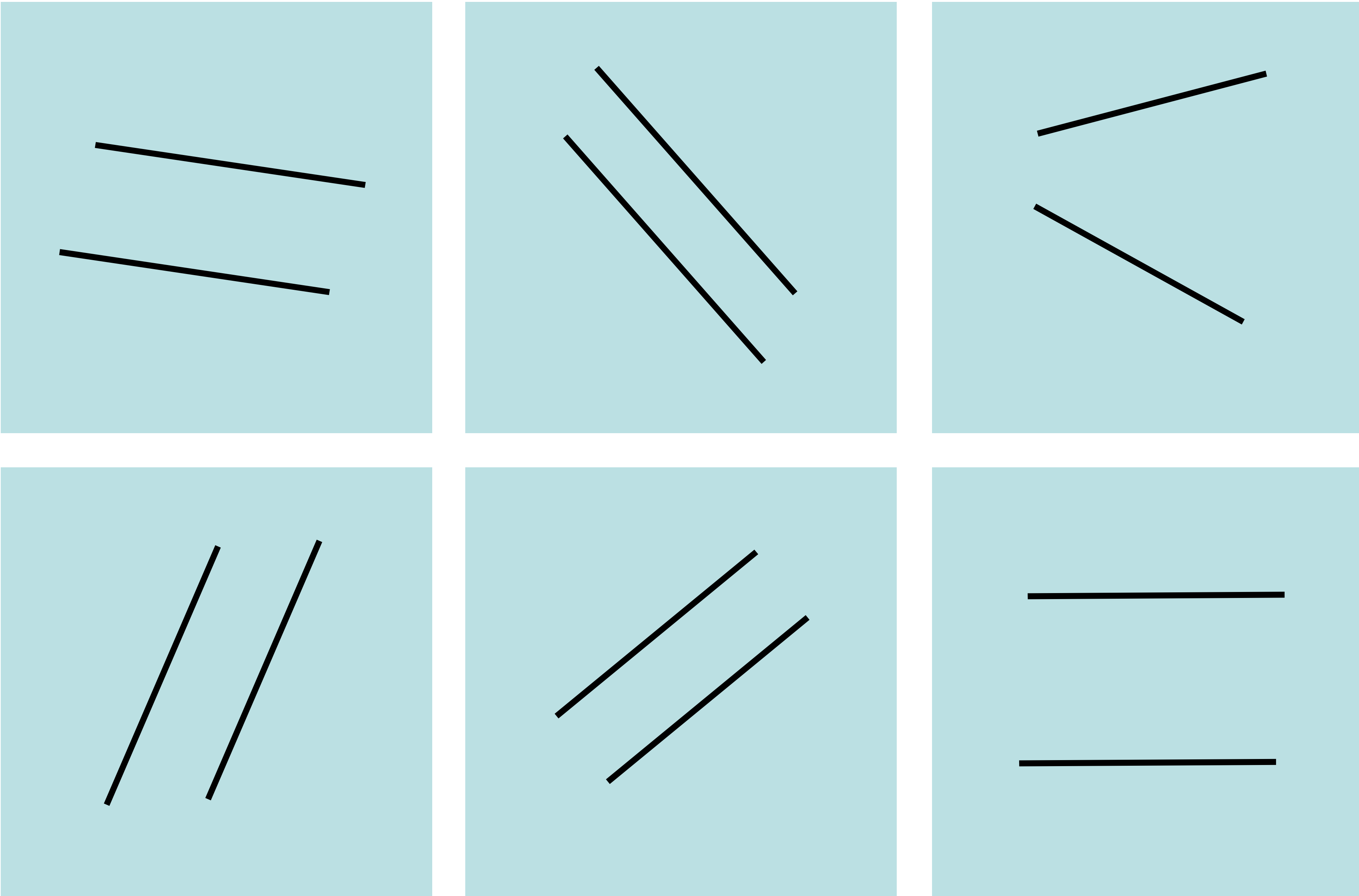
Grid cells

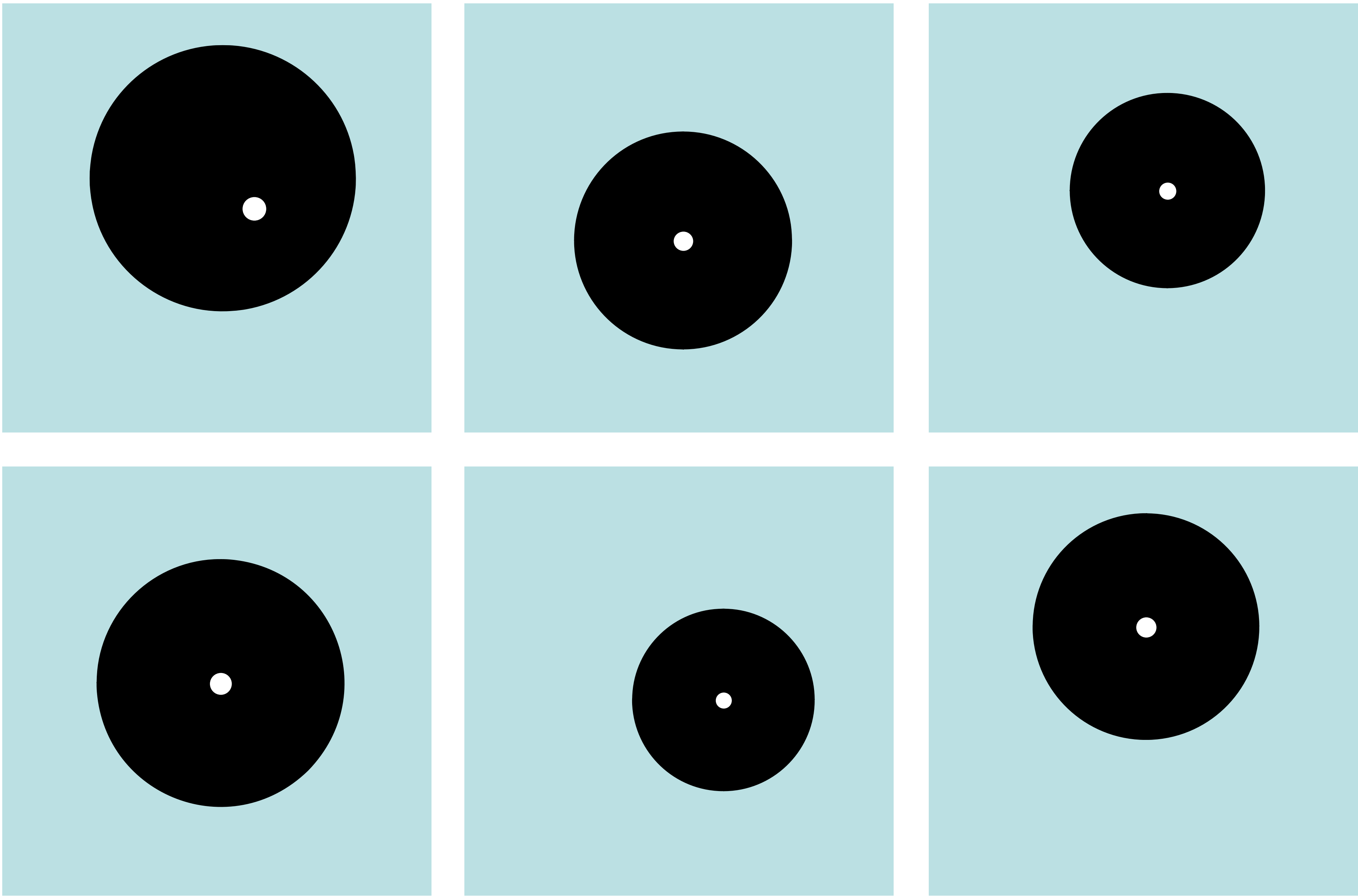


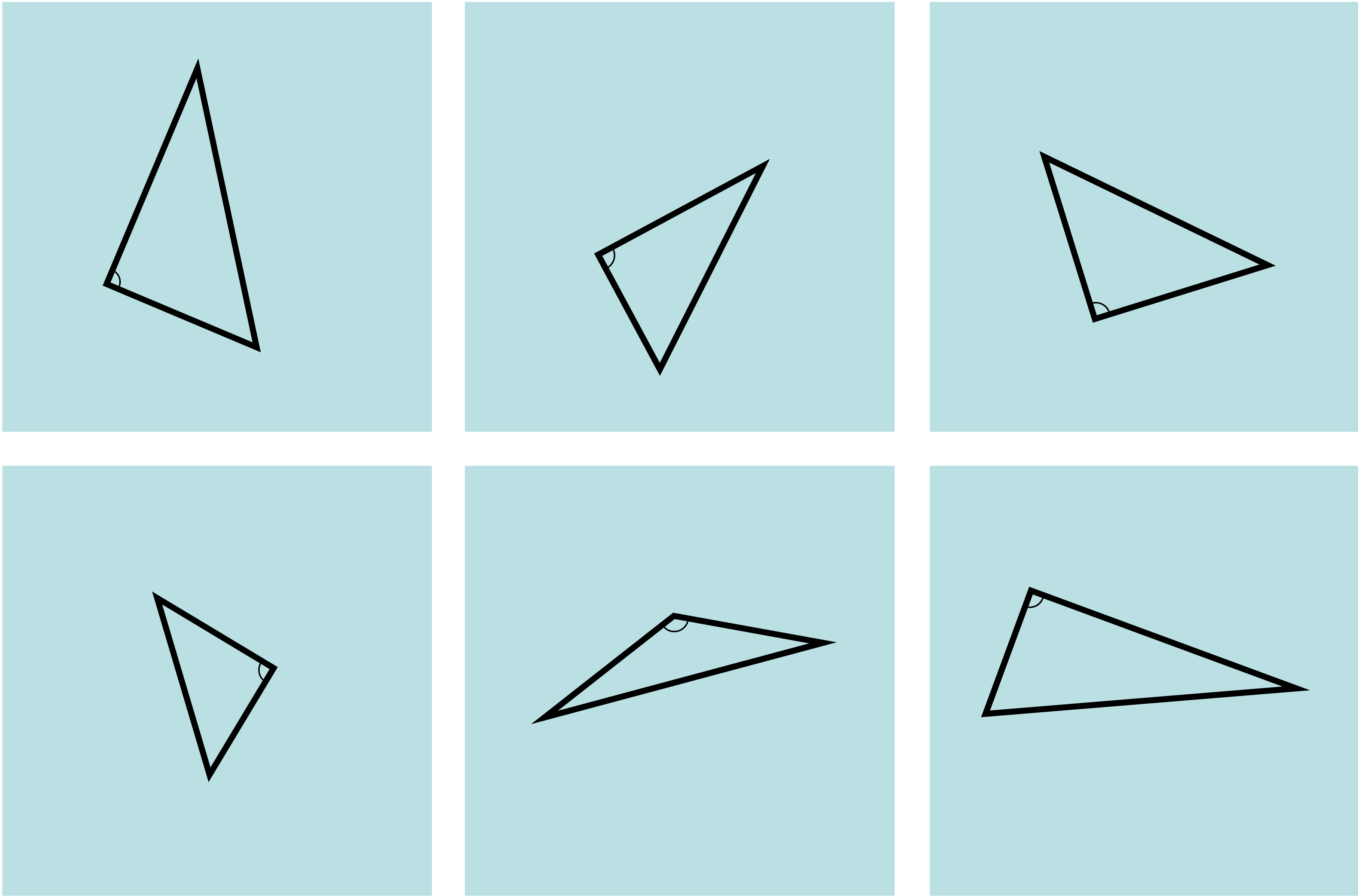
Stanislas Dehaene, Véronique Izard,
Pierre Pica, Elizabeth Spelke

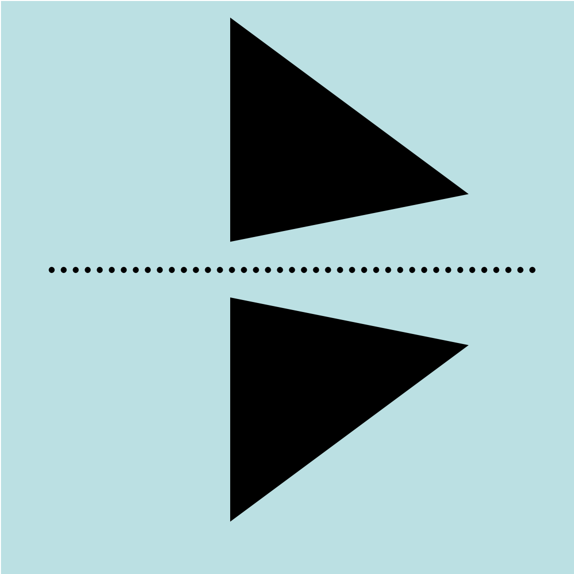
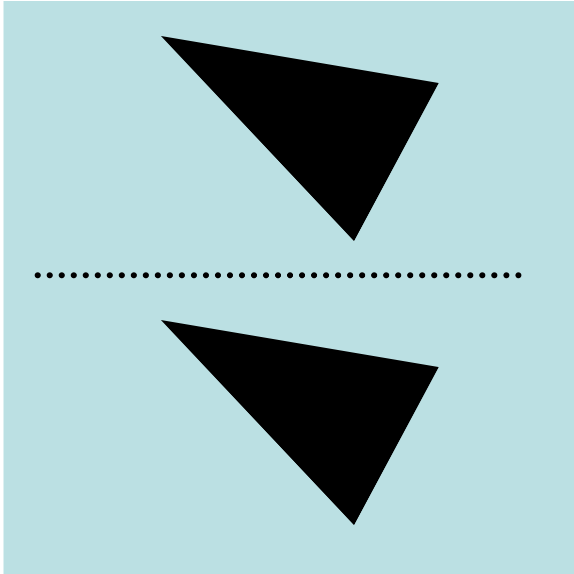
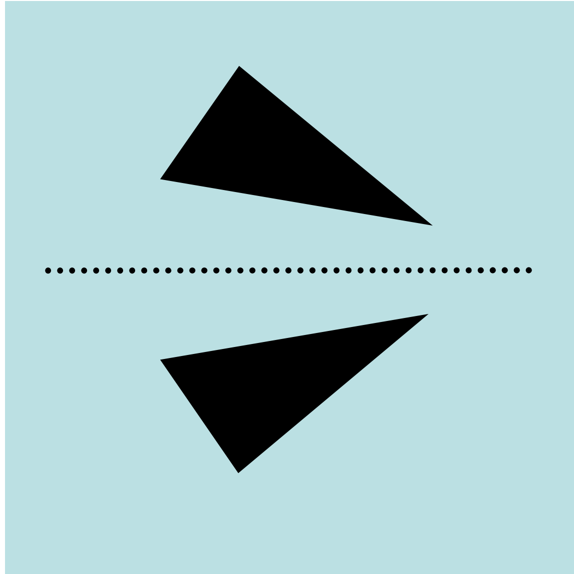
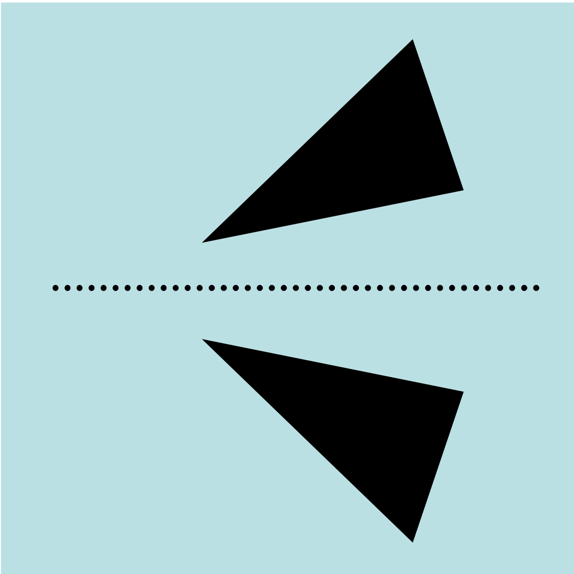
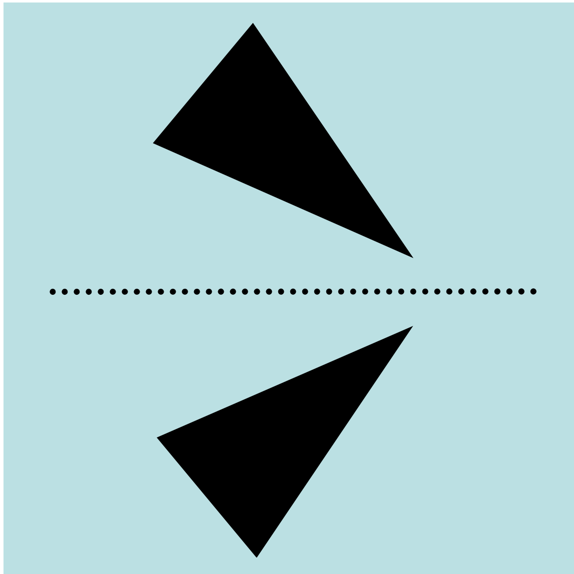
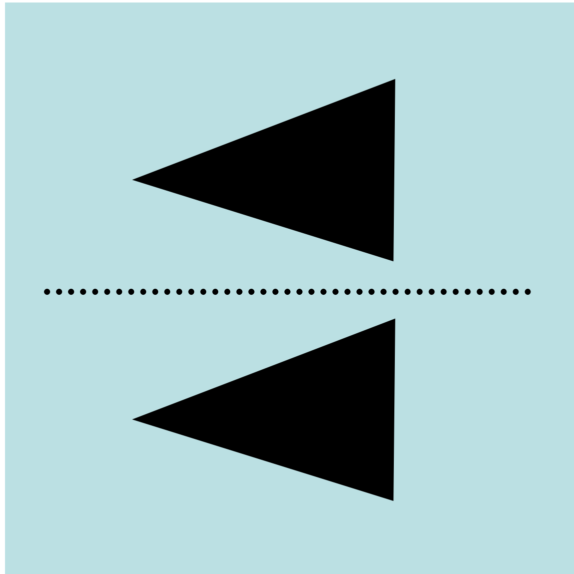
Core knowledge of geometry in an Amazonian indigene group

Science, January 2006

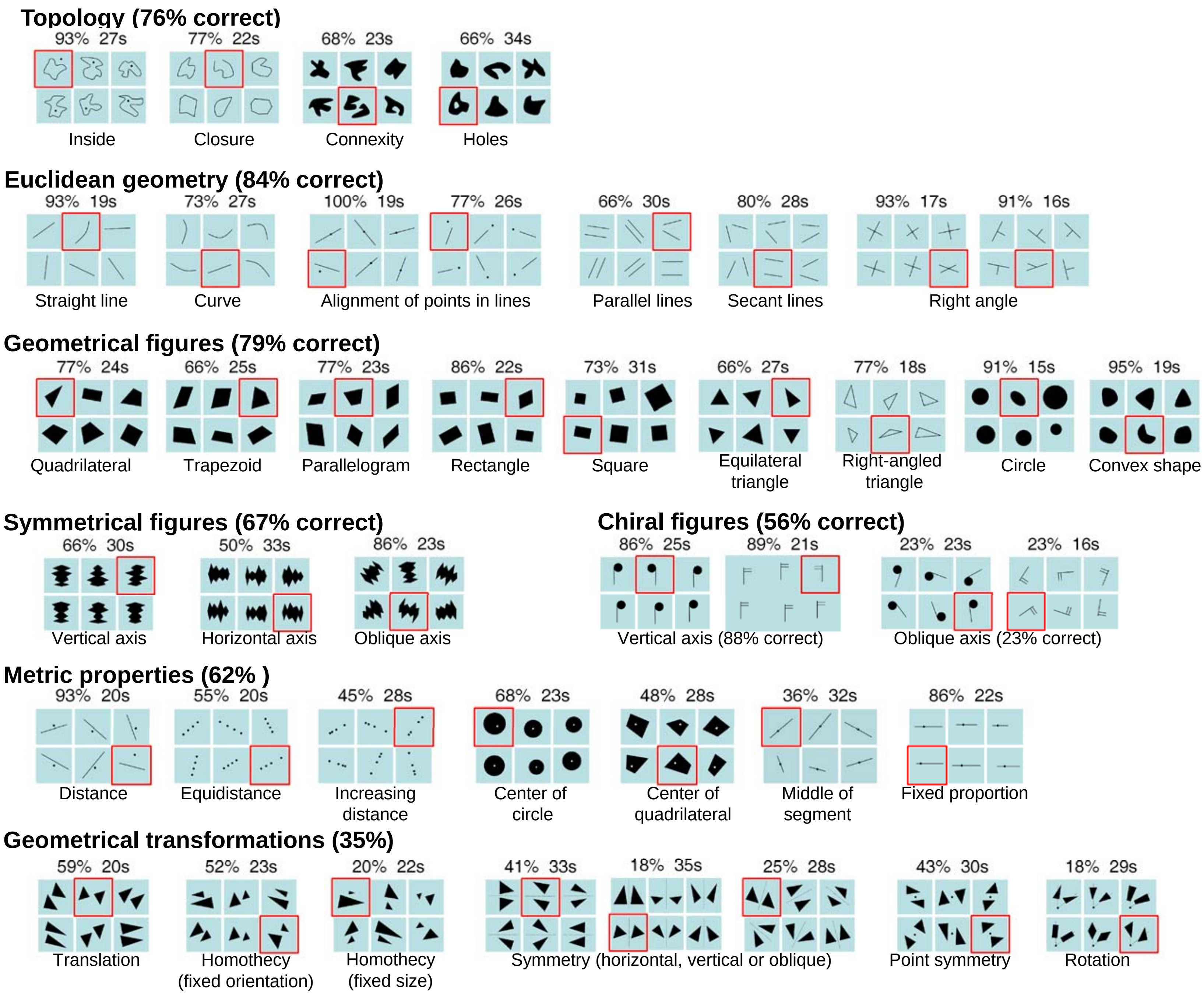






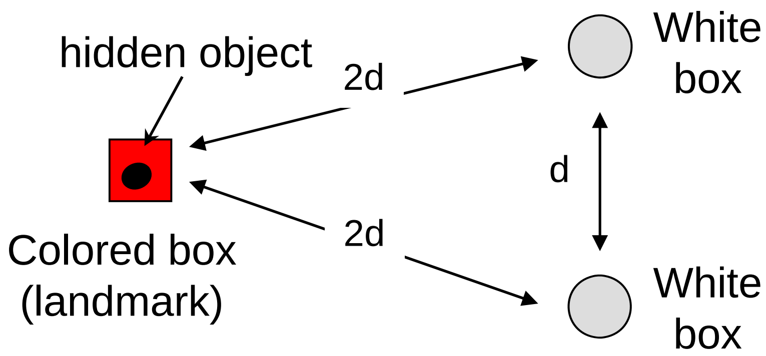







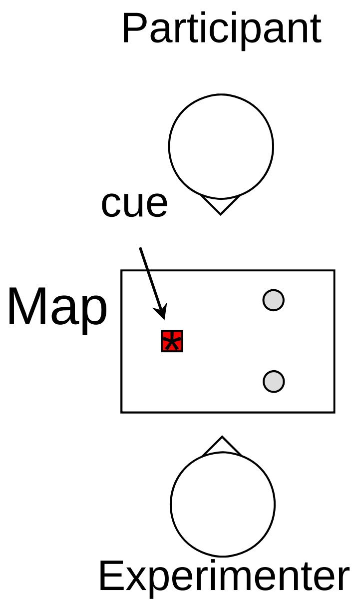
Core concepts of geometry are available to uneducated, monolingual Munduruku indians

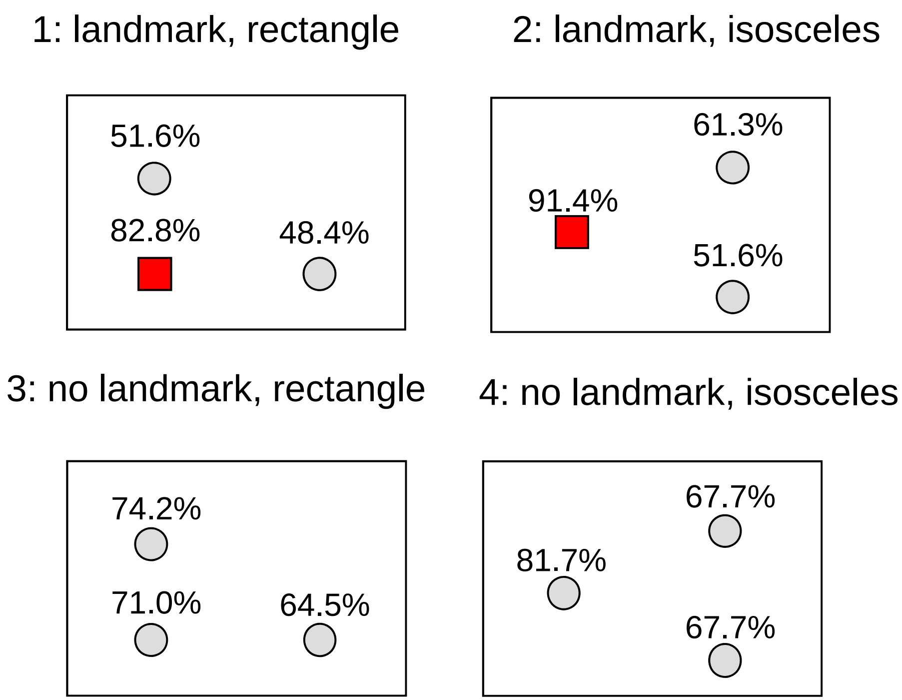


The Munduruku can use geometrical relations in a « map »

Environment

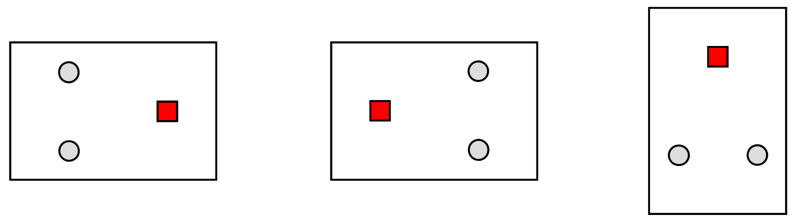




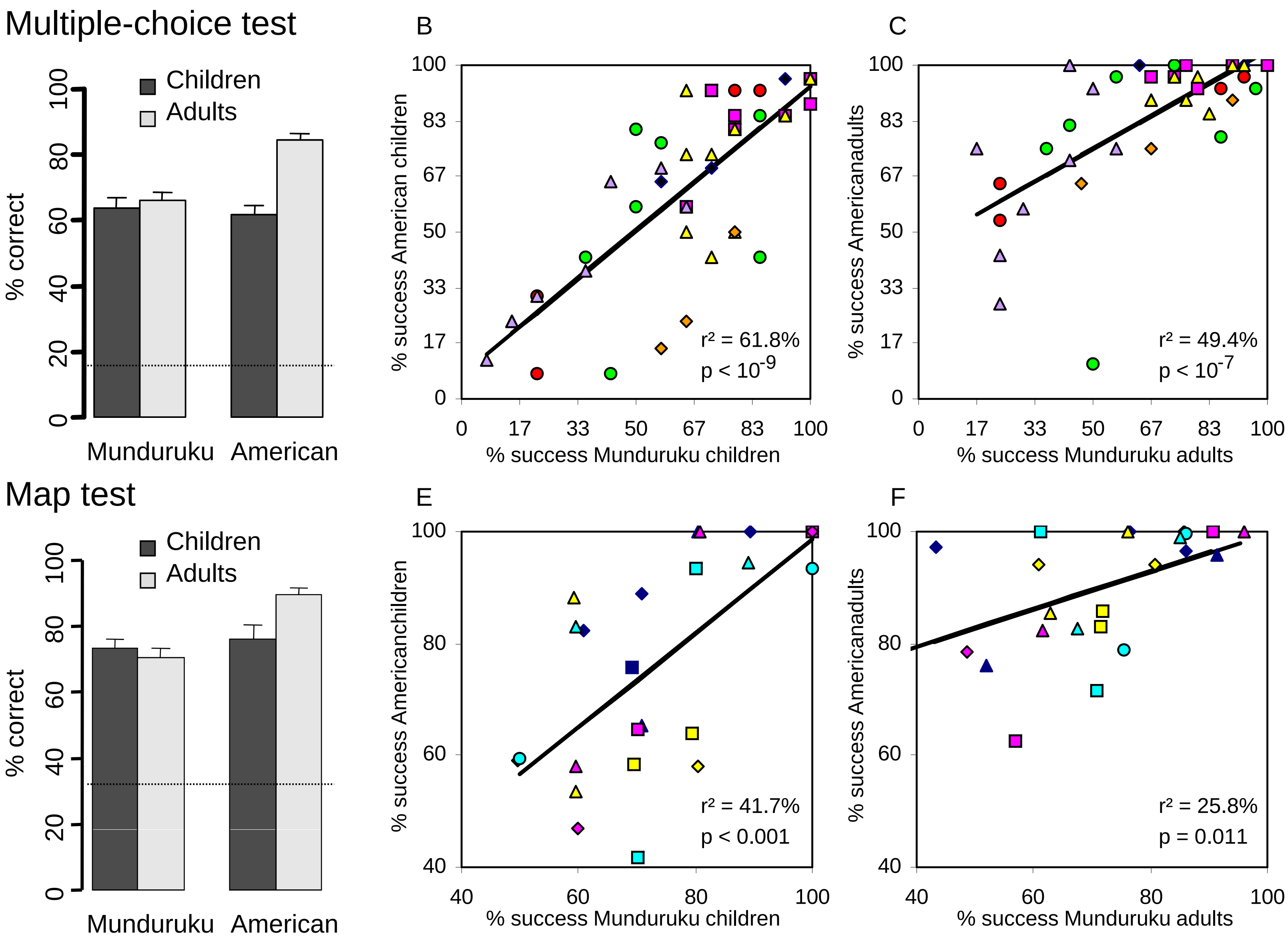


Success regardless of map orientation

Egocentric	Allocentric	Rotated
71.0%	70.6%	72.6%



The geometrical intuitions of Munduruku indians correlate with those of American children and adults



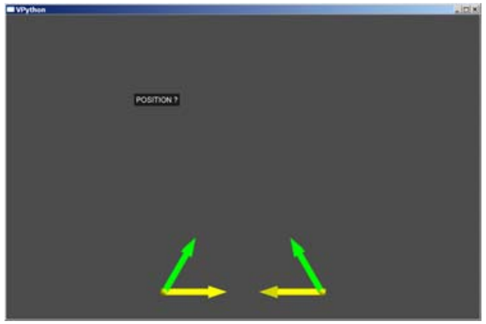
Can the Munduruku understand Euclidian and Non-Euclidian geometry?

Vivid description of a world with very small villages and very straight paths

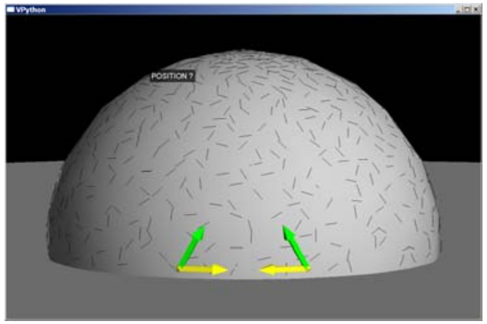
The world can be flat or spherical.

Task = find the third village, and indicate how the paths would meet there.

Plane trials



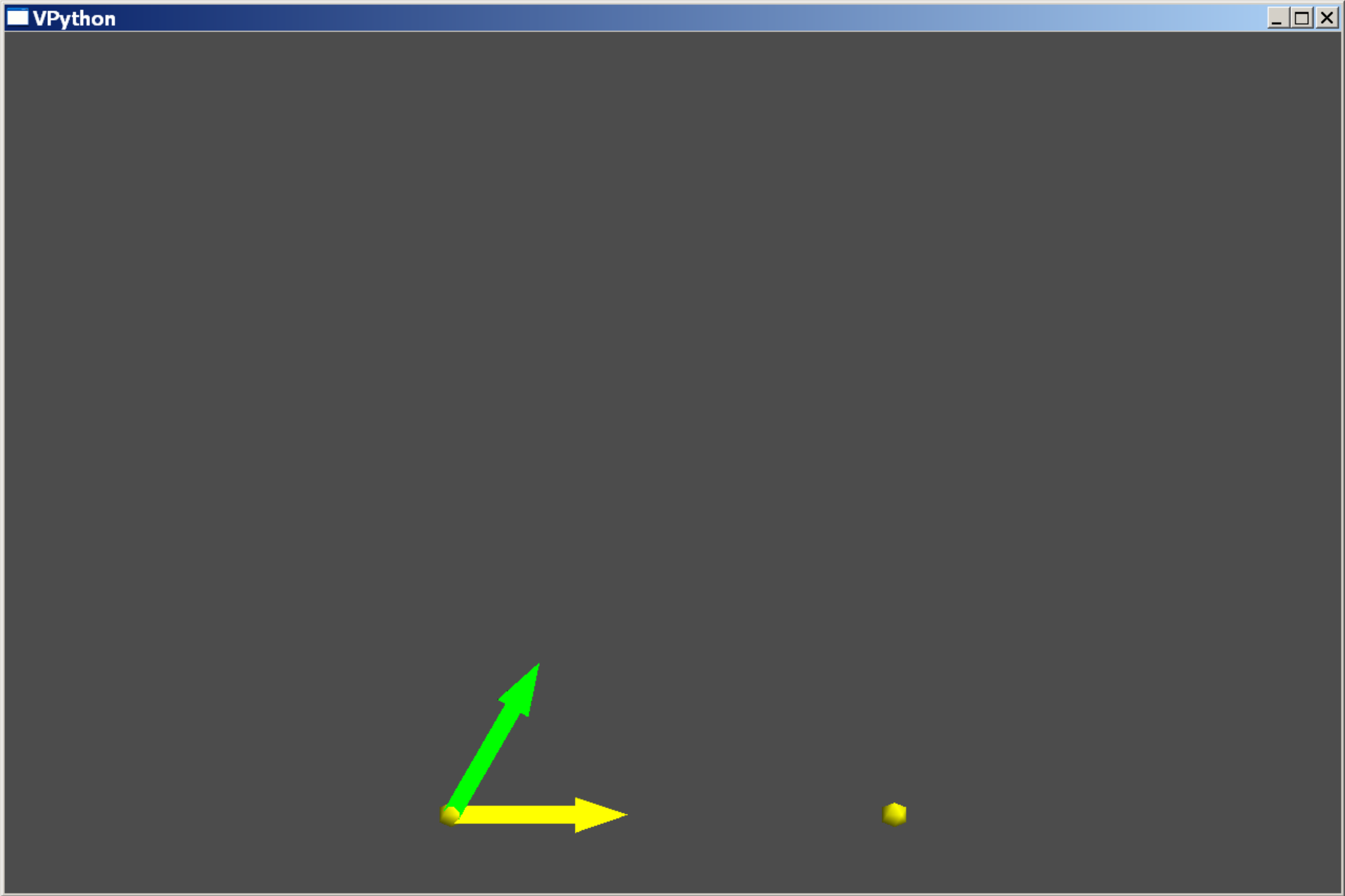
Sphere trials



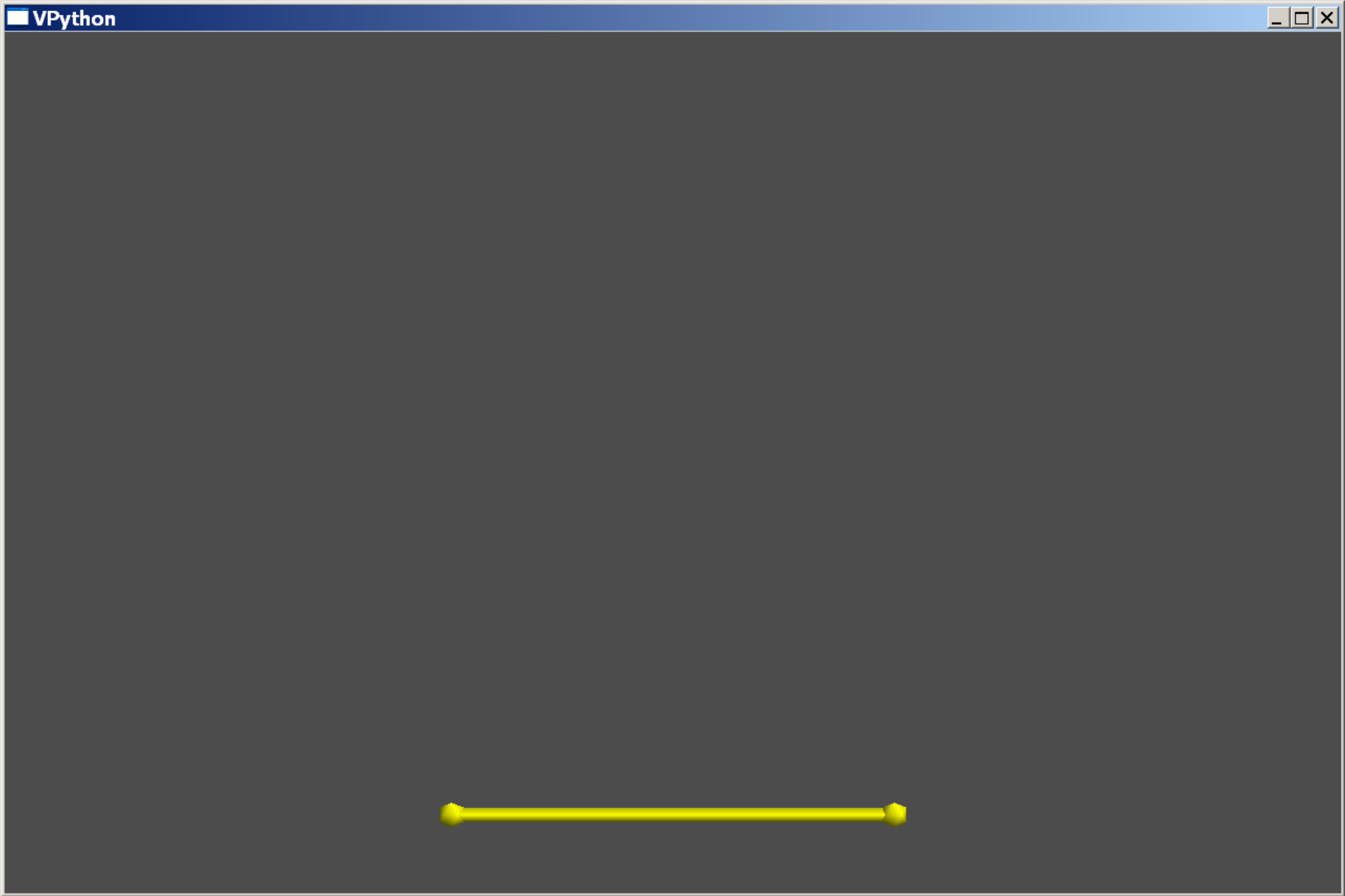
Two response modes

-indicate angle with the two hands (angle measured by the experimenter)

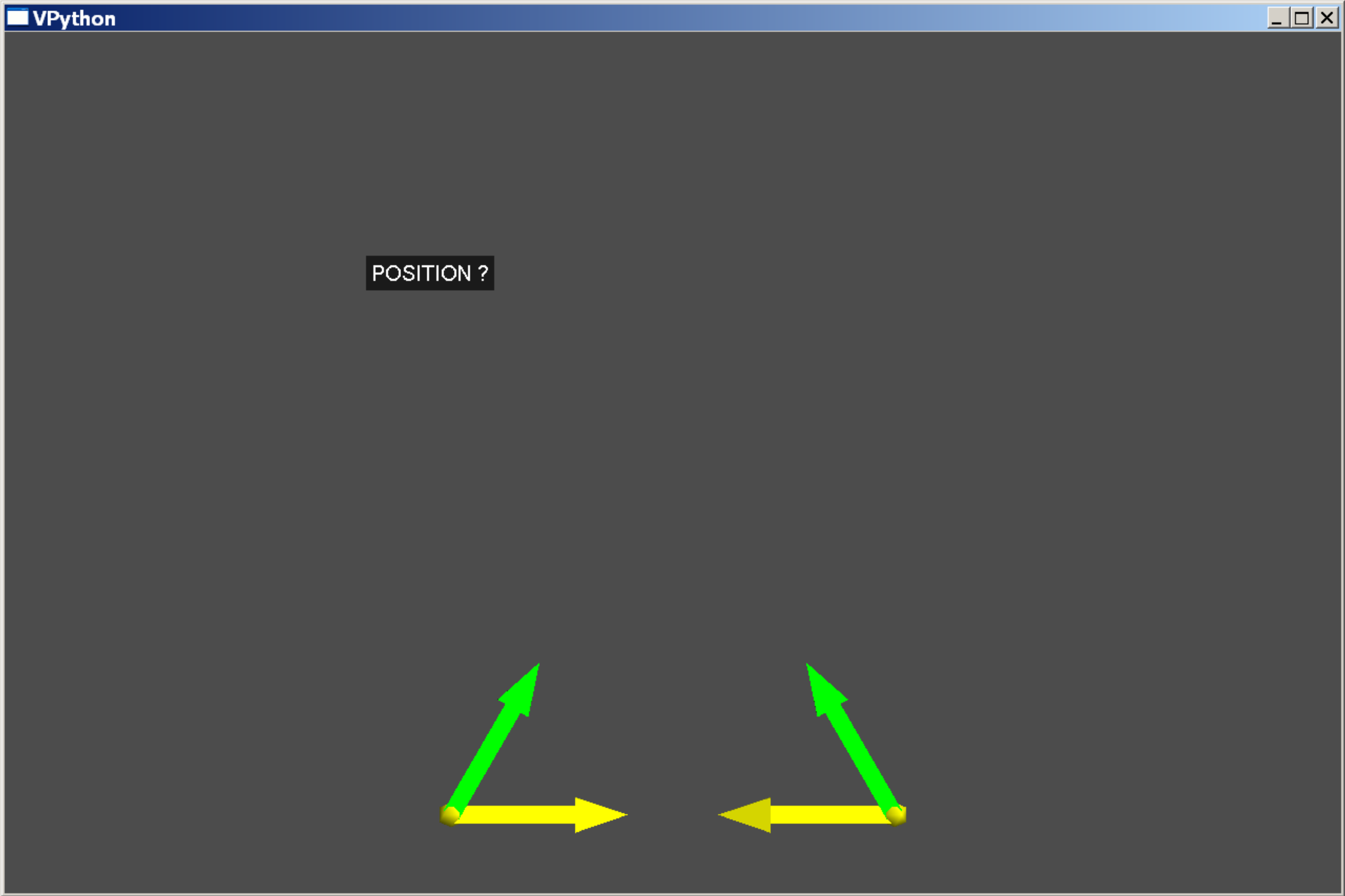
-indicate the angle directly by manipulating the goniometer



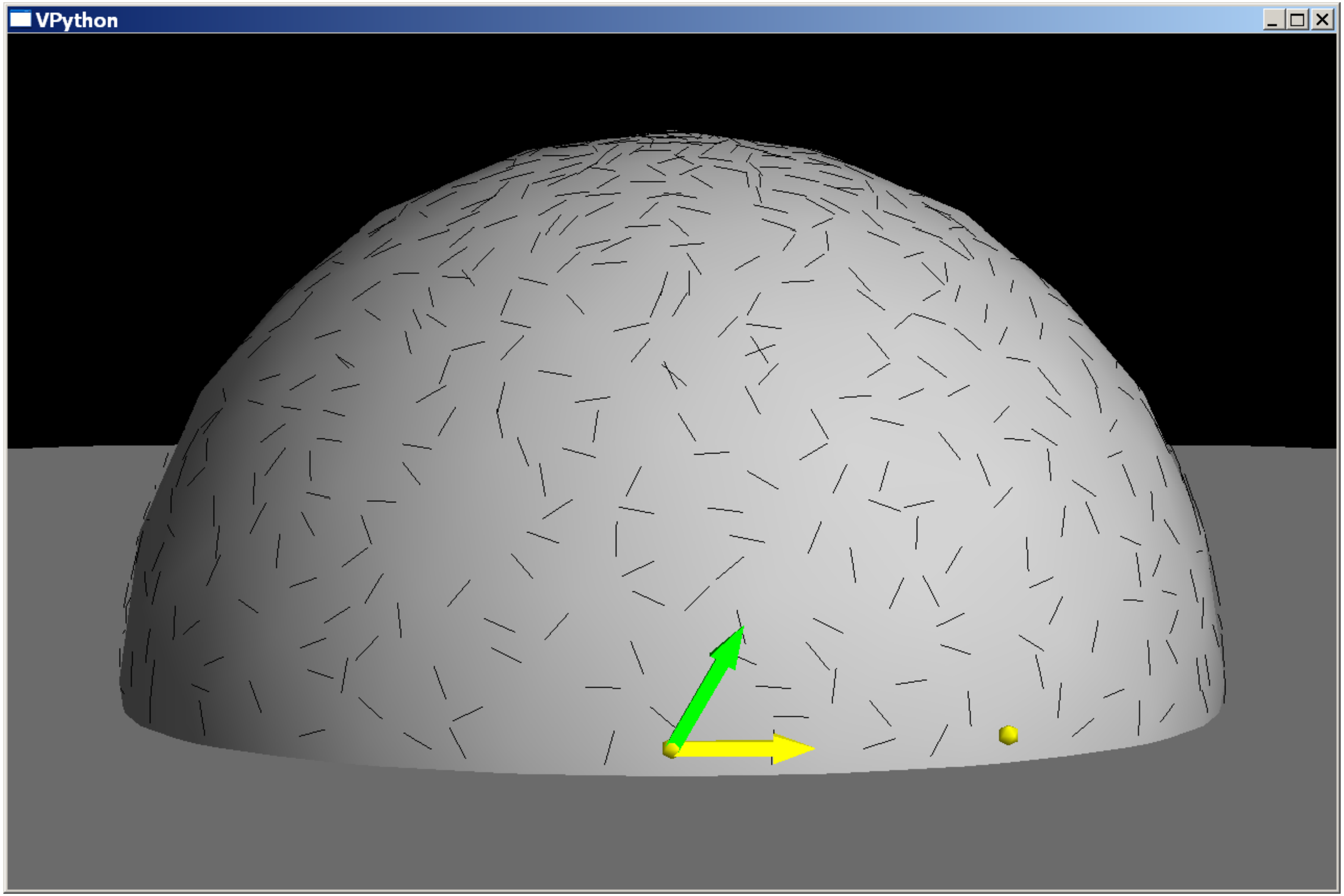
This is a place where the land is very flat.
You can see two villages. From this village here, you can see two paths.



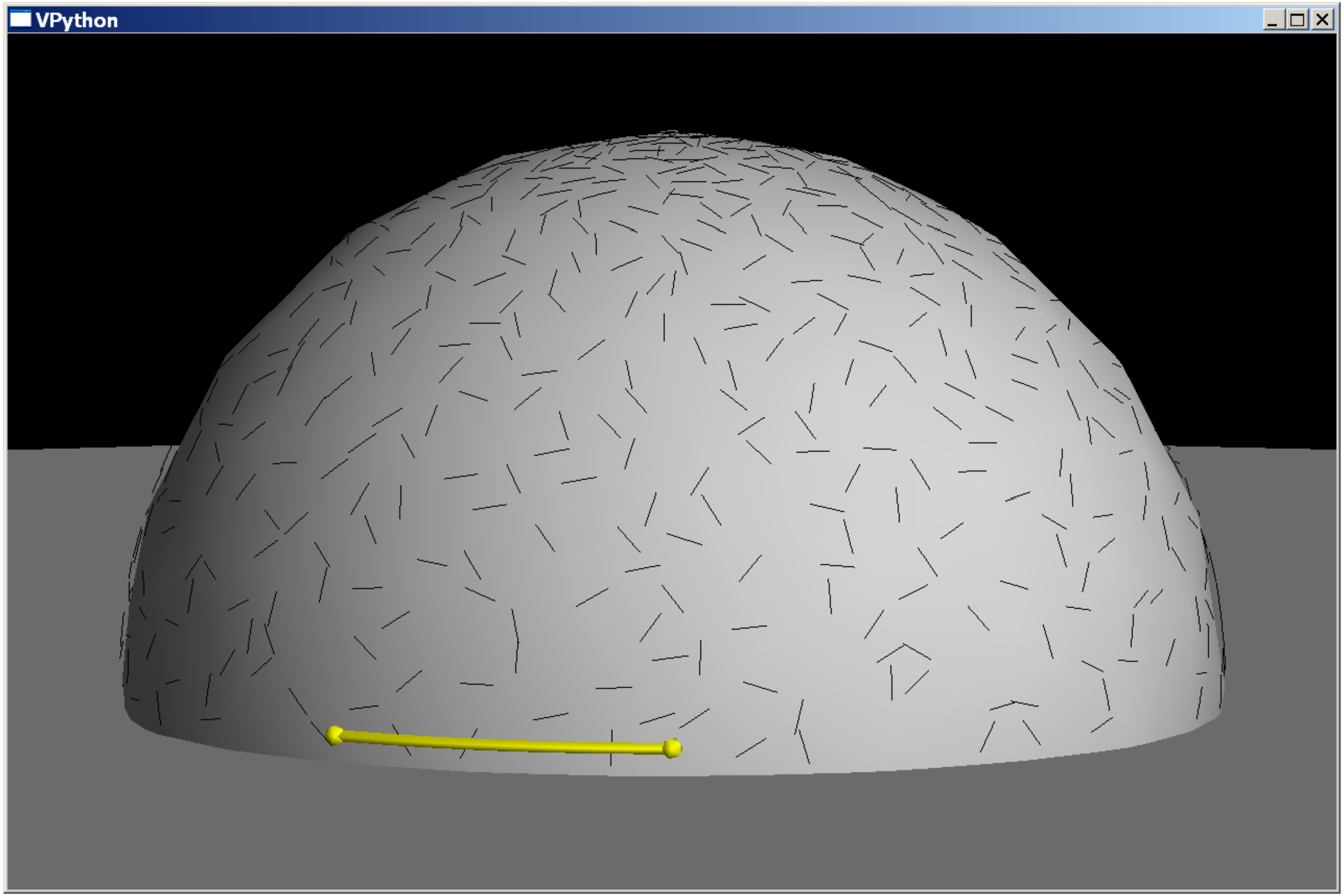
One of the paths leads straight to the other village.



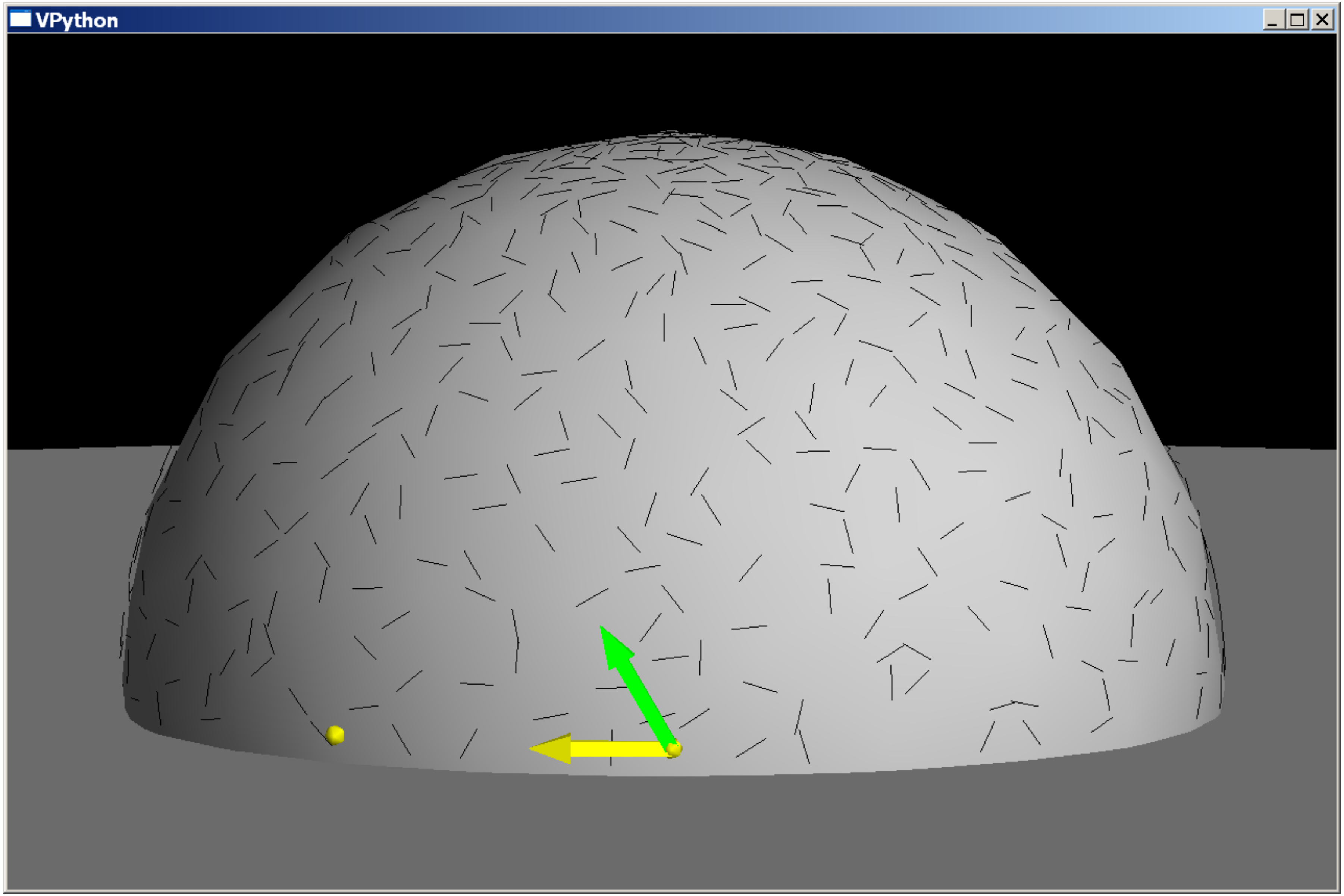
At the other village too, there are two paths. The two green paths go straight to another village. I would like to tell me where those two paths lead. Show me where is the other village. Also show me how the two paths look like at this village.



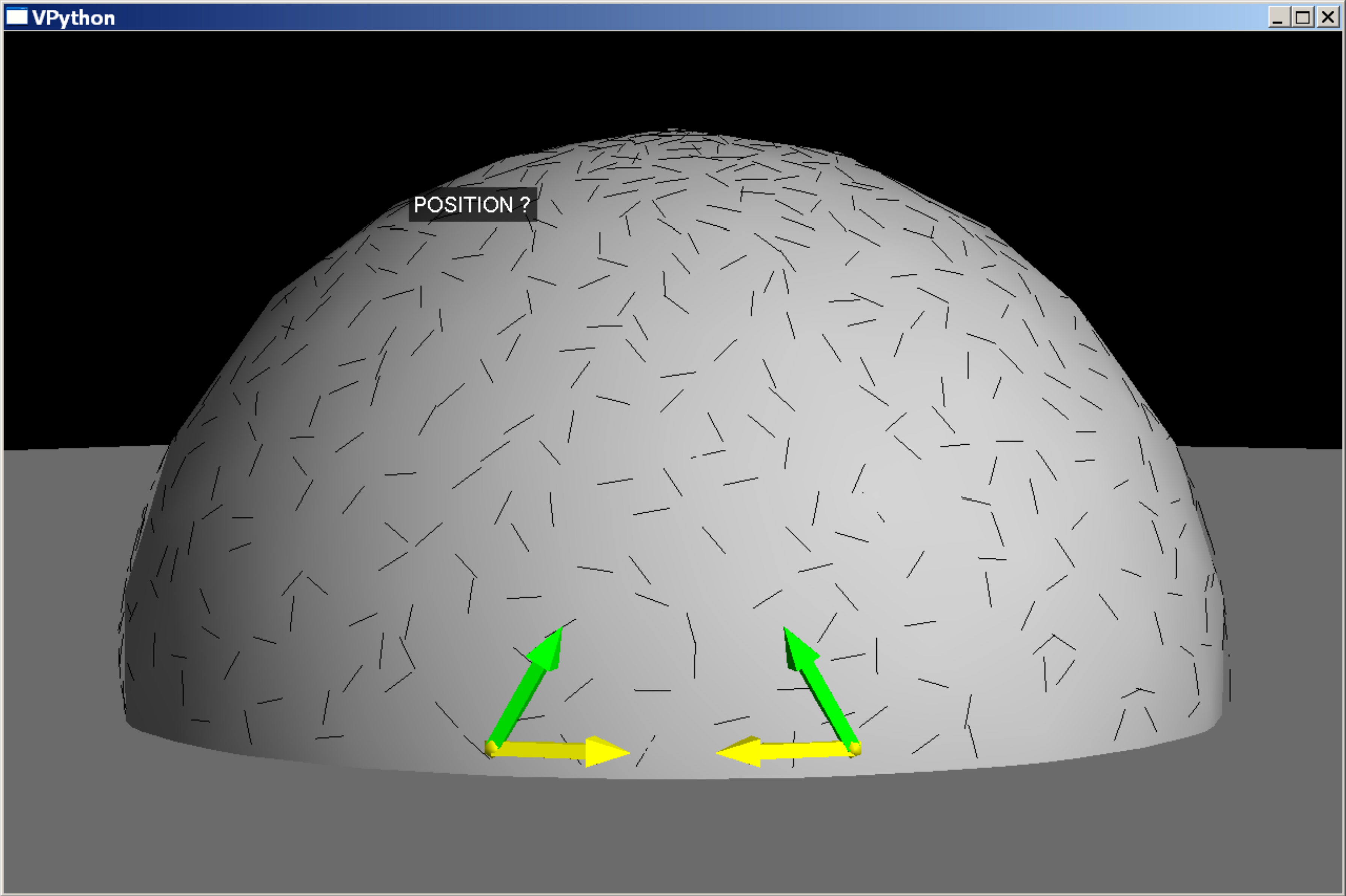
This is a place where the land is very curved and round.
You can see two villages. From this village here, you can see two paths.



One of the paths leads straight to the other village.

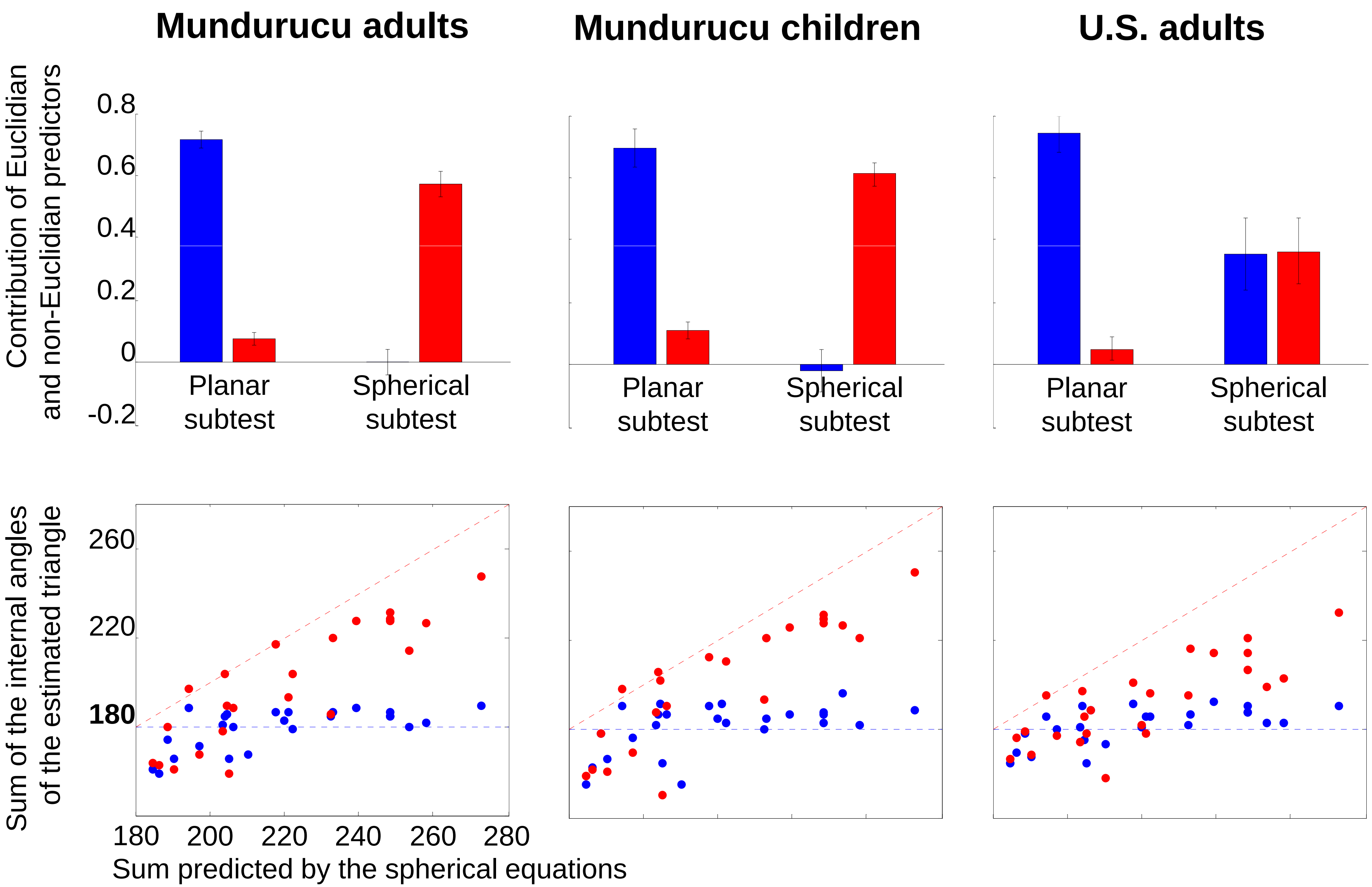


At this village too, there are two paths.



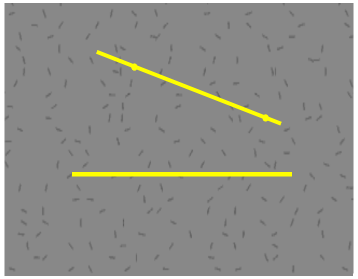
The two green paths go straight to another village. I would like to tell me where those two paths lead. Show me where is the other village. Also show me how the two paths look like at this village.

The Munduruku implicitly understand the sum of angles in both Euclidian and Non-Euclidian geometry



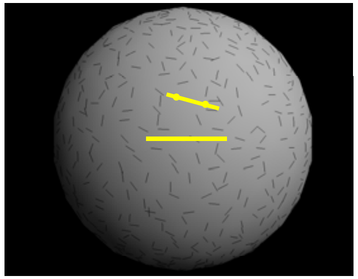
A “Platonic dialog” with the Mundurucu

Plane - Euclidian



“This is a place where the land is very flat, and goes on forever and ever …”

Sphere - NonEuclidian

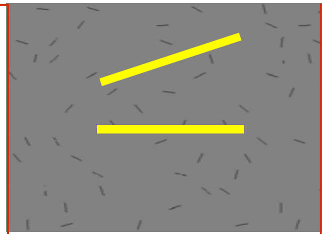


“This is a place where the land is very round, like a ball …”

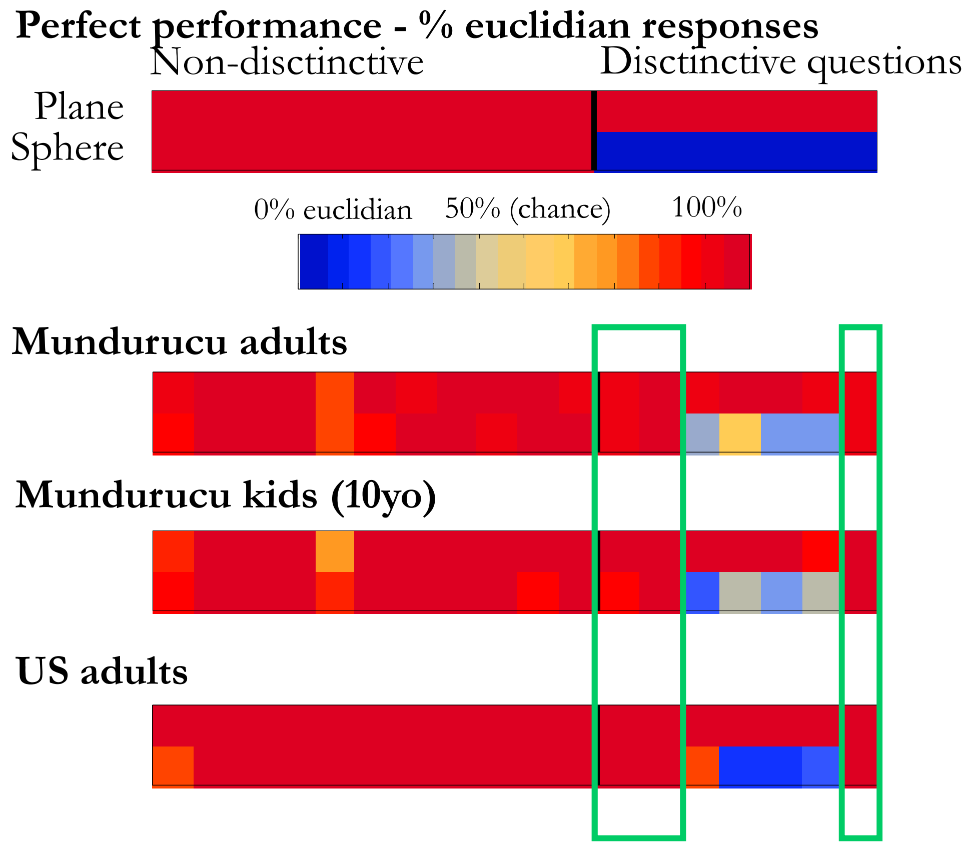
Intuitions of Euclidian geometry were essentially perfect.
All participants revised their intuitions in the non-Euclidian case, but failed to understand that there are no parallels on a sphere.

“Let us approach to see better …”

Are the paths going to meet on this side?
Eucl: Yes
NonEucl: Yes



Are the paths going to meet on this side?
Eucl: No
NonEucl: Yes



Conclusions

• Once presented with the appropriate '**mental model**', we all have intuitions of both Euclidean and non-Euclidean geometries.

• However, intuitions of Euclidean geometry seem to be more immediate